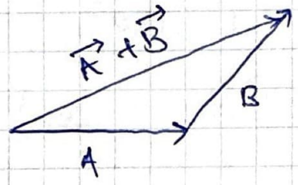
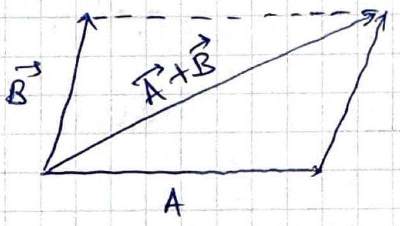
1. PRELIMINARY MATHEMATICS

1.1 INTRODUCTION

- velocity $(\vec{v}) = \dfrac{d\vec{x}}{dt}$ (in general $\dfrac{d\vec{r}}{dt}$)
- Accleration $(\vec{a}) = \dfrac{d\vec{v}}{dt} = \dfrac{d^2\vec{r}}{dt^2}$
- Triangle Law of Addition



- Parallelogram Law of Addition



- Uniform Circular Motion

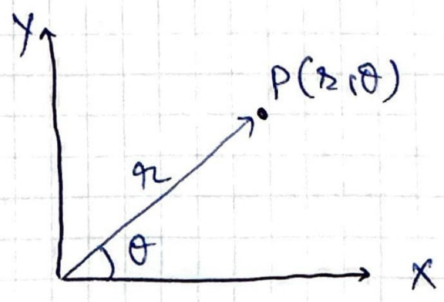
$$\underbrace{\theta}_{\text{Angular displacement}} = \underbrace{\omega}_{\text{Angular velocity}} \underbrace{t}_{\text{time}}$$

I. $\vec{r} = r\cos\omega t\,\hat{i} + r\sin\omega t\,\hat{j}$

II. $\vec{v} = \dfrac{d\vec{r}}{dt} = -r\omega\sin\omega t\,\hat{i} + r\omega\cos\omega t\,\hat{j}$
$= -r\omega(\sin\omega t\,\hat{i} - \cos\omega t\,\hat{j})$

III. $\vec{a} = \dfrac{d\vec{v}}{dt} = -\omega^2\vec{r}$

1.2 MOTION IN PLANE POLAR COORDINATES



$X = r\cos\theta$

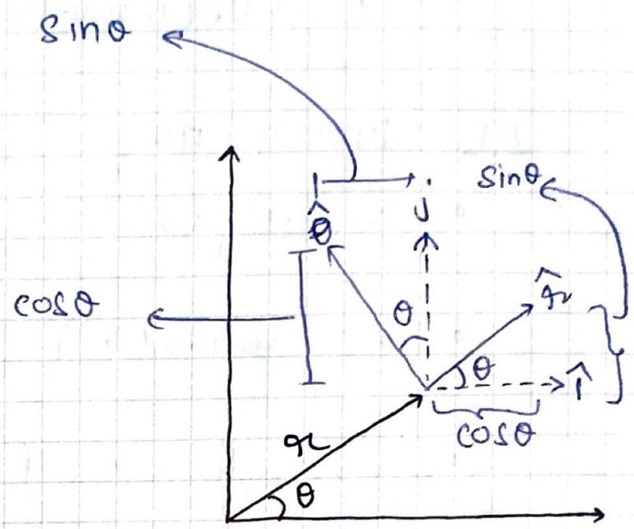
$Y = r\sin\theta$

But we want to study $\hat{r}$ & $\hat{\theta}$ direction too.

Hence,

$$\hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$$

$$\hat{\theta} = \sin\theta(-\hat{i}) + \cos\theta\,\hat{j}$$



additionally

$$\vec{r} = |\vec{r}|\cdot\hat{r}$$

$$\Rightarrow \vec{r} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$$

$$\Rightarrow \vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{dt}$$

Evaluating $\dfrac{d\hat{r}}{dt} = \dfrac{d}{dt}(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

$$= (-\sin\theta)\cdot\dot{\theta}.\hat{i} + \cos\theta\cdot\dot{\theta}\,\hat{j}$$

$$= \dot{\theta}\left[\sin\theta(-\hat{i}) + \cos\theta\,\hat{j}\right]$$

$$= \dot{\theta}\,\hat{\theta}$$

Hence, $\boxed{\dot{\hat{r}} = \dot{\theta}\,\hat{\theta}}$

Similarly $\boxed{\dfrac{d\hat{\theta}}{dt} = -\dot{\theta}\,\hat{r}}$

So $\vec{v} = \dfrac{d\vec{r}}{dt} = \dfrac{d}{dt}(r\cdot\hat{r})$

$$= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt}$$

$$= \dot{r}\hat{r} + r\cdot\dot{\theta}\,\hat{\theta}$$

Hence $\boxed{\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\,\hat{\theta}}$

Similarly $\vec{a} = \dfrac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta}$

$$+ r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$$

$$= (\ddot{r} - r\dot{\theta}^2)\,\hat{r}$$

$$+ (r\ddot{\theta} + 2\dot{r}\dot{\theta})\,\hat{\theta}$$

Hence $\boxed{\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}}$

Summary

I. $\dfrac{d\hat{r}}{dt} = \dot{\theta}\hat{\theta}$ $\qquad \hat{r} = \cos\theta\,\hat{i} + \sin\theta\,\hat{j}$

II. $\dfrac{d\hat{\theta}}{dt} = -\dot{\theta}\hat{r}$ $\qquad \hat{\theta} = \sin\theta(-\hat{i}) + \cos\theta\,\hat{j}$

III. $\vec{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

IV. $\vec{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

1.3 MATHEMATICAL APPRROXIAMATIONS

1.3.1 Binomial Series

- $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \dfrac{n(n-1)(n-2)}{3!}x^3 + \ldots$

  for $x \in (-1,1)$ for any value of n

  Note thet for integer n — Series terminates

- FOR $|x| > 1$,

  $(1+x)^n = x^n(1 + 1/x)^n = x^n(1 + n(1/x) + \ldots)$

  eg. $(1001)^{1/3} = (1000+1)^{1/3}$

  $= (1000^{1/3})\left(1 + \dfrac{1}{1000}\right)^{1/3}$

  $= 10\left(1 + \dfrac{1}{3}\cdot\dfrac{1}{1000}\right)$

  $= 10(1 + 1/3000)$

  $= 10(1 + 0.3\times 10^{-3})$

  $= 10(1.0003)$

  $(1001)^{1/3} = 10.003$

1.3.2 Taylor's Series

Any function could be represented by power Series.

$$f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \ldots$$

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} a_n x^n$$

hence for $x = 0$, we have $f(0) = a_0$

and taking differentiation

$$f'(x) = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \ldots$$

at $x = 0$, we should get $f'(x) = a_1$

and for kth derivative we have

$$a_k = \frac{1}{k!} f^{(k)}(x) \Big|_{x=0}$$

and finally for constant in power series expansion we could write:-

$$f(x) = f(0) + f'(0) \cdot x + f''(0) \frac{x^2}{2!} + f'''(0) \frac{x^3}{3!} + \ldots$$

this series allow us to find good approxiamation to f(x) for small x (near zero)

we could further generelize to have:-

$$f(a+x) = f(a) + f'(a) x + f''(a) \frac{x^2}{2!} + f'''(a) \frac{x^3}{3!} + \ldots$$

and another variation is:-

$$f(t) = f(a) + f'(a) \cdot (t-a) + f''(a) \frac{(t-a)^2}{2!} + \frac{f'''(a)(t-a)^3}{3!} + \ldots$$

using Taylor Series we could write :-

1. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots$

II. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^6 - \frac{1}{6!}x^8 + \cdots$

III. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

2. MECHANICS OF A PARTICLE

2.1 Basics

- Equation of motion of a particle

$$m\frac{d^2\vec{r}}{dt} = \vec{F}$$

- work done to bring particle at 2 from 1 :-

$$W_{12} = \int_1^2 \vec{F}\cdot d\vec{r}$$

or $$W_{12} = \underbrace{T_2}_{\text{kinetic energy at 2}} - \overbrace{T_1}^{\text{kinetic energy at 1}}$$

- conservative Force

$$\oint \vec{F}\cdot d\vec{r} = 0$$

or $$\vec{\nabla}\times\vec{F} = 0$$ Many PYQ have been asked

if curl of $\vec{F}$ is zero, then some scaller function should exist to satisfy $\vec{F} = -\vec{\nabla}V(\vec{r})$

Thus, $$W_{12} = \int_1^2 \vec{F}(\vec{r})\, d\vec{r}$$

$$= -\int_1^2 \vec{\nabla}V\cdot d\vec{r}$$

$$= -(V_2 - V_1)$$

$$\boxed{W_{12} = V_1 - V_2}$$

Note in conservative force field, total mechanicel energy is independent of position of particle and is constand of motion.

Note that in spherical form :-

$$\vec{F} = -\vec{\nabla} V = -\left[\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}\hat{\phi}\right]$$

- Additionally, Impulse is :-

$$\int_{t_1}^{t_2} \vec{F}\,dt = \int_{t_1}^{t_2} \frac{d\vec{p}}{dt}\,dt$$

$$= \vec{P_2} - \vec{P_1}$$

So impulse ~~in~~ is change of momentum

- Angular Momentum $\vec{L} = \vec{r} \times \vec{P}$

Angular " (or Torque) $= \vec{r} \times \vec{F}$

Hence Torque $(\vec{\tau}) = \frac{d\vec{L}}{dt}$

2.2 Momentum

$$\int dp = \int F\,dt$$

Consider a system of N interacting particles, with mass $m_1, m_2, m_3 \ldots m_N$.

Position of jth particle is $\vec{r_j}$.

Equation of motion of jth particle is

$$\cancel{f_j} \quad \vec{f_j} = \frac{d\vec{P_j}}{dt}$$

or $$\vec{f_j}^{\,int} + \vec{f_j}^{\,ext} = \frac{d\vec{p_j}}{dt}$$

writing the same expression for all the particles we could have $\sum f_j^{int} + \sum f_j^{ext} = \sum \frac{d\vec{p_j}}{dt}$

Now, we know $\vec{F}_{12} = -\vec{F}_{21}$

So all the term involving internal force would cancel out each other. Hence.

$$\sum f_j^{ext} = \sum \frac{d\vec{p}_j}{dt}$$

$$\Rightarrow \quad \vec{F}_{ext} = \frac{d\vec{p}}{dt}$$

total momentum

Hence external force applied is rate of change of momentum of system.

Center of Mass

As we have $\vec{F} = \frac{d\vec{p}}{dt}$ $\left\{ \text{Note } \vec{p} = \sum \vec{p}_j \right\}$ $= \sum m_j \dot{r}_j$

So, as there are N particle in the system, thus we could assume that there is some balance point $\vec{R}$ where all the masses are centered.

Thus

$$M\ddot{\vec{R}} = \frac{d\vec{p}}{dt} = \sum m_j \ddot{r}_j$$

$$\Rightarrow \quad \ddot{\vec{R}} = \frac{\sum m_j \ddot{r}_j}{M}$$

$$\vec{R} = \frac{1}{M} \sum m_j r_j$$

mass of rth particle

Position vector of center of Mass

total mass of system

mass of jth particle

Thus if System has N particle, $m_1, m_2, m_3 \ldots m_N$ at $\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots \vec{r}_N$ then it could be thought that their mass at balanced at $\vec{R}$ given by

$$\vec{R} = \frac{1}{M} \sum m_j r_j$$

And for continous mass –

$$\vec{R} = \frac{1}{M}\int r\,dm \quad \left\{ \text{by using } \vec{R} = \frac{1}{M}\sum_{j=1}^{\infty} m_j \vec{x}_j \right\}$$

with limit $N \to \infty$

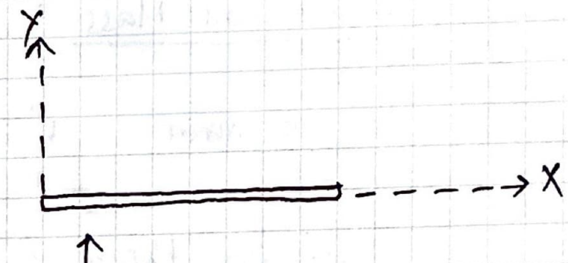
and $dm = \rho dv$

<u>Question</u>

Find center of Mass of a Non-Uniform Rod.

A rod has length L has a non-uniform density λ (mass/length) and $\lambda = \lambda_0 (S/L)$ where λ_0 is a constant, S is distance marked from a end.

Find Center of Mass.



Rod Kept on X-Axis

We have

$$R = \frac{1}{M}\int x\,dm$$

We need M and dm.

~~M = d length~~

$$\cancel{M = \text{mass per unit length} \times \text{total length}}$$

$$\cancel{= \lambda_0 \cdot \frac{S}{L} \times L}$$

$$\cancel{M = \lambda_0 S}$$

{ Because non-uniform mass }

$$M = \int dm = \text{mass per unit length} \times \text{total length}$$

$$= \int_0^L \lambda\,dx$$

$$= \int_0^L \lambda_0 \cdot \frac{S}{L}\,dx$$

$$= \int_0^L \lambda_0 \cdot \frac{x}{L} \cdot dx$$

{ as S is distance marked from one end, So it is X for us here as we have put rod on X-Axis }

$$M = \frac{1}{2}\lambda_0 L$$

So $\vec{R} = \frac{1}{M}\int r.dm$

$$= \frac{1}{M}\int_L r.\lambda dl$$

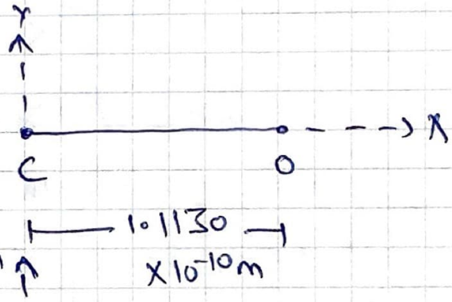
$$= \frac{2}{\lambda_0 L}\int_0 (x\hat{i}).\frac{\lambda_0 x}{L}.dx$$

$$\vec{R} = \frac{2}{3}L\hat{i}$$

PYQ. I. Distance b/w center of ~~ca~~ carbon & oxygen atoms in the carbon Monoxide gas Molecule is 1.130×10^{-10}m. Locate center of Mass relative to carbon atom

II. Find the center of Mass of a homogenous Semi-circular plate of radius a.

Answer



I. $\vec{R} = \frac{\sum m_i \vec{r_i}}{M}$

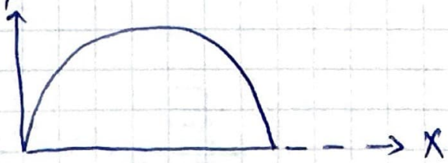
$$= \frac{12\,amu \times 0 + 16 \times 1.1130\times10^{-10}m\,\hat{i}}{28\,amu}$$

$$= \frac{16}{28}\times 1.130\times10^{-10}\,m\,\hat{i}$$

$= 6.448\times10^{-11}$ m toward the oxygen atom from carbon atom

So C.O.M of CO Molecule is 6.448×10^{-11} m from carbon atom toward oxygen atom

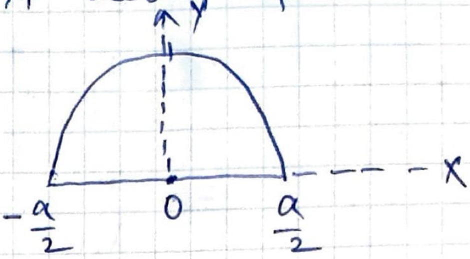
II. Let us place the semi circular plate on XY plane.



So diameter is along the X-Axis

then we could say the C.O.M will be on a point $(a/2, Y)$.

To reduce the mathemetical difficulty, we could place the plate such that it become Symmetric to Y Axis.



So center of Mass Coordinates would be Zero along X Axis and $\bar{y}$ along Y Axis given by

$$\bar{y} = \frac{1}{A}\iint y\, dA$$

where A is Area $= (1/2)\pi a^2$

from knowledge of polar coordinates

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$dA = dr . r d\theta = r dr d\theta .$$

$$\text{So, } \bar{y} = \frac{1}{A}\iint r\sin\theta . dr . r d\theta$$

limits are : $r \in [-a/2, a/2]$

$\theta \in [0, \pi]$

Hence
$$\bar{y} = \frac{1}{A}\int_{-a/2}^{a/2} r^2 dr \int_0^{\pi} \sin\theta d\theta$$

$$= \frac{2}{\pi a^2}\left[r^3/3\right]_{-a/2}^{a/2}\left[-\cos\theta\right]_0^{\pi}$$

$$= \frac{4a}{3\pi}$$

Therefore Coordinates of C.O.M are at $(0, 4a/3\pi)$.

Momentum & Flow of Mass

Generally, $\vec{Force} = \frac{d\vec{P}}{dt}$

$= M\frac{d\vec{v}}{dt}$

and mass is constant.

But in some cases mass is not constant like Motion of Rocket/Spacecraft etc.

Suppose a spacecraft is moving in space at velocity $\vec{v}$.

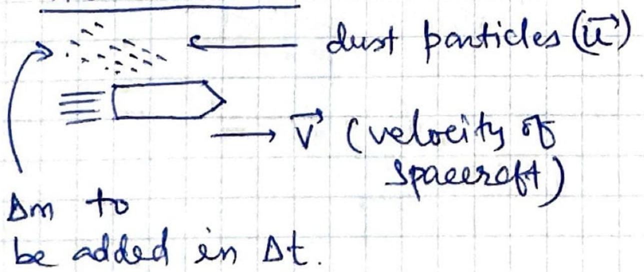
~~The~~ Then Spacecraft encounter stream of dust at rate dm/dt which gets embeded to its body.

Dust particles had velocity $\vec{u}$ just before it hit the space craft.

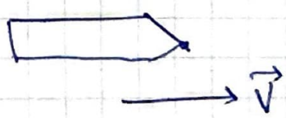
So if dust particles are getting embeded then mass of Rocket will increase and its speed would decrease. Therefore to maintain the same speed Spacecraft will have to apply more thrust i.e. Force.

We will see case when time is $\underline{t}$ and $\underline{t + \Delta t}$.

At time t



At time $t + \Delta t$



Mass of ~~Rocket~~ ~~(M(t))~~ Spacecraft System $(M(t))$.

Mass of Spacecraft $M(t) + \Delta m$

So, Momentum at time t.

$$P(t) = M(t)\cdot\vec{v} + \Delta m\cdot\vec{u}$$

Momentum at time $t + \Delta t$

$$P(t+\Delta t) = \left(M(t) + \Delta m\right)\vec{V}$$

So change in Momentum is ΔP.

$$\Delta P = P(t+\Delta t) - P(t)$$

$$= \left[M(t) + \Delta m\right]\vec{V} - M(t)\vec{V} - \Delta m \cdot \vec{u}$$

$$\Delta P = \Delta m\,(\vec{V} - \vec{u})$$

or $$\frac{\Delta P}{\Delta t} = (\vec{V} - \vec{u})\,\frac{\Delta m}{\Delta t}$$

or $$\vec{F} = (\vec{V} - \vec{u})\,\frac{dm}{dt}$$

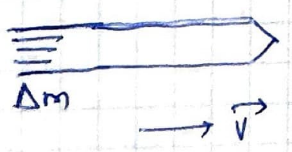
So $\vec{F}$ is required external force to keep spacecraft in motion.

<u>Motion of Rocket</u>

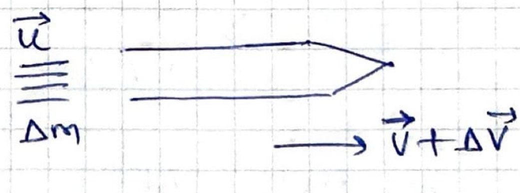
Consider Rocket at time t.

Between time t and $t + \Delta t$ a fuel of mass Δm is burnt, and expelled as gas to provide thrust.

Speed of gas is $\vec{u}$ relative to rocket.



<u>At time t</u>



$$V_{ER} = V_E - V_R$$

$$\vec{u} = V_E - (\vec{V} + \Delta\vec{V})$$

$$\vec{V_E} = \vec{V} + \Delta\vec{V} + \vec{u}$$

$\hookrightarrow$ velocity of exhaust relative to observer

Momentum of System at t :-

$$\vec{P}(t) = (M + \Delta m) \cdot \vec{V}$$

Momentum of System at $t + \Delta t$:-

$$\vec{P}(t + \Delta t) = M(\vec{V} + \Delta\vec{V}) + \Delta m(\vec{V} + \Delta\vec{V} + \vec{u})$$

Hence change in momentum is :-

$$\Delta\vec{P} = \vec{P}(t + \Delta t) - \vec{P}(t)$$

$$= M\vec{V} + M\Delta\vec{V} + \Delta m\vec{V} + \Delta m \cdot \Delta\vec{V} + \Delta m \cdot \vec{u} - M\vec{V} - \Delta m\vec{V}$$

$$\Rightarrow \Delta\vec{P} = M\Delta\vec{V} + \Delta m \cdot \vec{u}$$ {we neglected $\Delta m \cdot \Delta\vec{V}$}

$$\Rightarrow \frac{dp}{dt} = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t}$$

$$\Rightarrow \vec{F} = M\frac{d\vec{v}}{dt} + \vec{u}\frac{dm}{dt}$$ {$\vec{u}$ is defined to be positive in direction of $\vec{V}$}

[In real Scenario if $\vec{v}$ is positive $\vec{u}$ is negative]

~~Hence~~ $$\vec{F} = M\frac{d\vec{v}}{dt} - u\frac{dM}{dt}$$ {as $\frac{dm}{dt} = \frac{-dM}{dt}$}

Now if external force $\vec{F}$ is zero it is case of Rocket in Free Space.

Hence,
$$0 = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt}$$

$$\Rightarrow M\frac{d\vec{v}}{dt} = \vec{u}\frac{dM}{dt}$$

$$\Rightarrow d\vec{v} = \vec{u}\frac{dM}{M}$$

$$\Rightarrow V_f - V_i = \vec{u}\left[\ln M_f - \ln M_i\right]$$

or $$V_f - V_i = -\vec{u}\left(\ln \frac{M_i}{M_f}\right)$$

Rocket is gravitational field is :-

$$\vec{F} = M\vec{g} = M\frac{d\vec{v}}{dt} - \vec{u}\frac{dM}{dt}$$

$$\Rightarrow M(\vec{g} - d\vec{v}/dt) = -\vec{u}\, dM/dt$$

$$\Rightarrow \vec{g}\, dt - dv = -\vec{u}\frac{dM}{M}$$

$$\Rightarrow d\vec{v} = \vec{g}.dt + \vec{u}\,(dM/M)$$

$$\Rightarrow V_f - V_i = \vec{g}(t_f - t_i) + \left(-\vec{u} \ln \frac{M_i}{M_f}\right)$$

$$\Rightarrow V_f = u \ln(M_i/M_f) - g.t_f$$

took positive velocity upward.

Summary

I. Eqn. of motion of Rocket in Free Space

$$V_f - V_i = -\vec{u} \ln\left(\frac{M_i}{M_f}\right)$$

$\vec{u}$ is relative to Rocket

II. Eqn. of motion of Rocket in Gravity

$$V_f = u\ln(M_i/M_f) - gt$$