3. WORK AND ENERGY

3.1 INTRODUCTION

When Force is applied then energy is spent and thet converts into work. (Sometime even when force is applied - work is zero)

$$\oint_C \vec{F}_{conservative} \cdot d\vec{r} = 0$$

for ~~se~~ conservative Force, work done over closed path is zero.

3.2 EQUATION OF MOTION IN 1D

Recall, $m\frac{d^2\vec{r}}{dt} = \vec{F}$

$\Rightarrow m\frac{d\vec{v}}{dt} = \vec{F}$ {we assumed 1D along X-Axis}

$\Rightarrow m\frac{d\vec{v}}{dt} \cdot dx = \vec{F} \cdot dx$

$\Rightarrow m\int \frac{dv}{dt} dx = \int F dx$

$\Rightarrow m\int \frac{dv}{dt} \cdot v dt = \int F dx$

$\Rightarrow m\int_{t_a}^{t_b} \frac{d}{dt}\left(\frac{1}{2}v^2\right) dt = \int_{x_a}^{x_b} F dx$

$\Rightarrow \frac{mV_b^2}{2} - \frac{mV_a^2}{2} = \int_{x_a}^{x_b} F(x) dx$

Hence, $\frac{1}{2}mV_b^2 - \frac{1}{2}mV_a^2 = \int_a^b F(x) dx$

is work Energy theorem as L.H.S is change in kinetic energy

3.2 APPLYING WORK-ENERGY THEOREM

$$W_{ba} = \oint_a^b \vec{F}.d\vec{r}$$

work energy theorem is merely statement that the change in kinetic energy is equal to the net work done.

For many forces of our interest, the work integral does not depend on particular path but only on end points. Such forces are called conservative force.

Escape velocity

Force on mass m is

$$F = -\frac{GM_em}{r^2}\hat{r}$$

$$= -mg\frac{R_e^2}{r^2}\hat{r}$$

Mathematical setup :-

$$\int_{R_e}^{r} \vec{F}.d\vec{r} = -mgR_e^2\int_{R_e}^{r}\frac{dr}{r^2}$$

$$\Rightarrow \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -mgR_e\int_{R_e}^{r}\frac{dr}{r^2}$$

$$\Rightarrow \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2 = -mgR_e\left[\frac{1}{R_e} - \frac{1}{r}\right]$$

- and if v_i has to be so large that mass m goes to infinity (escape velocity case) then $r \to \infty$ and $v \to 0$.

Hence

$$-\frac{1}{2}mv_i^2 = -mgR_e^2\left(\frac{1}{R_e}\right)$$

or $$v_i^2 = \cancel{mg}\ 2gR_e$$

or $$v_i = \sqrt{2gR_e} = \text{Escape velocity.}$$

Hence, Escape velocity is :-

$$\boxed{V_e = 2gR_e}$$

3.3 POTENTIAL ENERGY

Recall, for conservative force :-

$$\vec{F} = -\vec{\nabla}V(r)$$

or $$\int_a^b \vec{F}\cdot d\vec{r} = -\int \vec{\nabla}V(r)\cdot d\vec{r}$$

or $$W = V(a) - V(b)$$

or $$K.E_b - KE_a = V(a) - V(b)$$

or $$KE_a + V(a) = KE_b + V(b)$$

Sum of kinetic energy = Sum of Potential Energy

So, we have :-

I. $$V_b - V_a = -\int_a^b \vec{F}\cdot d\vec{r}$$

II. $$KE_i + V_i = KE_f + V_f$$

III. $$\vec{F} = -\vec{\nabla}V(r) \quad \{\text{conservative Force}\}$$

Potential Energy of Inverse Square Law

Centeral Force is a Force which depends upon the distance b/w both the masses. i.e. $\vec{F} \propto f(r)$. Precisely, we could write $\vec{F} = f(r) \cdot \hat{r}$

One such example is gravitational force, where $\vec{F} = \frac{G m_1 m_2}{r^2} \hat{r}$

As it is conservative force so a scaler function V must exist to satisfy $\vec{F} = -\vec{\nabla} V$.

Thus $V = \vec{F} \cdot d\vec{r}$ {V at ∞ is 0}

Additionally for inverse square law $\vec{F} \propto \frac{1}{r^2} \hat{r}$, we have potential of form $V(r) = \frac{A}{r}$.

3.4 What Potential Energy Tells about Force

Recall, $V_b - V_a = -\int_a^b \vec{F} \cdot d\vec{r}$

So any change in Potential might help to find the Force involved.

For one dimensional case, $\vec{F} = -\vec{\nabla} V$ gives

$$\boxed{F = -\frac{dV}{dx}.}$$

This is important result which tells us about nature of Force.

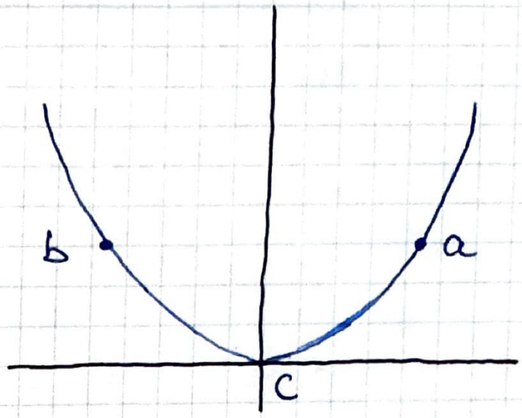
Potential Energy in case of Simple Harmonic oscillator is parabolic & given by $V = \frac{1}{2} kx^2$.

At point a $\rightarrow \frac{dv}{dx} > 0$

hence F is negative.

At point b $\rightarrow \frac{dv}{dx} < 0$

hence F is positive



And at c $\rightarrow \frac{dv}{dx} = 0$

and force vanishes

Notice that c is point of equilibrium and is right described by $\frac{dv}{dx} = 0$.

However if this occurs at maximum of U, the equilibrium is not stable.

3.5 ENERGY DIAGRAMS

Recall Energy = KE + PE

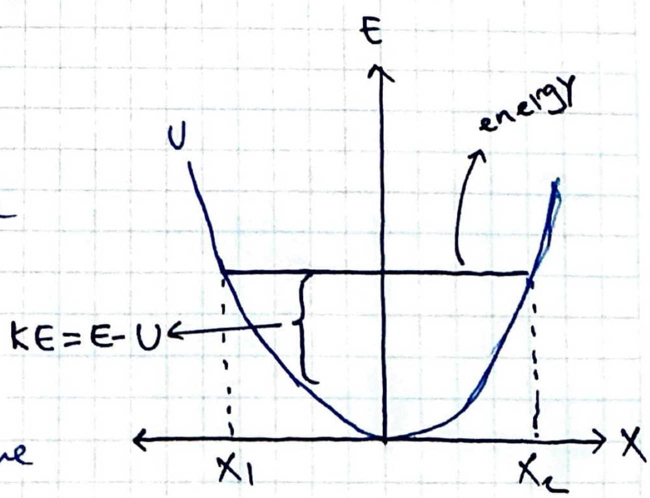
And as kinetic energy can not be negative, so motion of particle is bound when the potential energy is alway lesser than or equal to Energy. $V \leq E$

Study of Harmonic oscillator

As E increases $\rightarrow$ particle move towards extremes.

As E decreases $\rightarrow$ particle approaches mean position

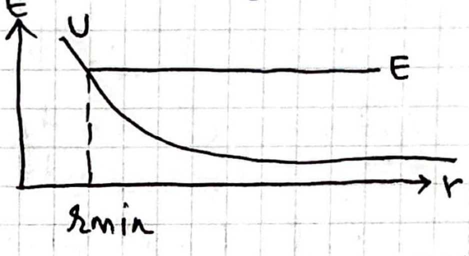
Motion is confined where $E \geq U$. At X_1 & X_2 we have zero kinetic energy.



Study of Repulsive Inverse Square Law

If $\vec{F} \propto \frac{1}{r^2}\hat{r}$ The $U \propto \frac{1}{r}$.

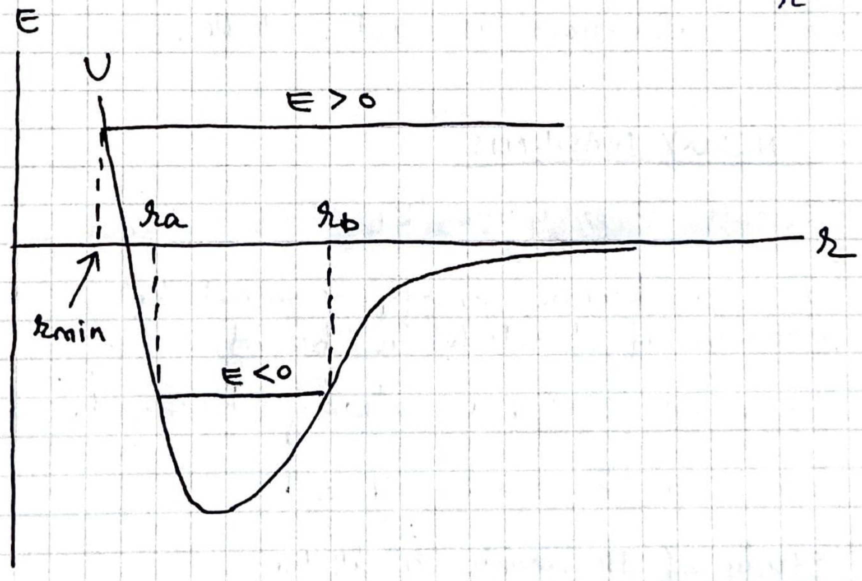
Here as U decrease with increasing r hence the motion can not be unbounded for large r.



At r_{min} Kinetic Energy comes at rest, but at $r \to \infty$, particle is unbound.

Study of Van waals force

Van der Waal Force is the case where $F \propto \frac{1}{r^7}$



In such ~~is~~ case if energy is negative, we always do have bound motion

For attractive force, U decreases with decreasing r.

For smaller interatomic distance, Force is always repulsive. Hence U increase rapidly with decreasing r.

From r_a to r_b $\longrightarrow$ motion is bounded.

3.6 CONSERVATION LAWS AND PARTICLE COLISSION

Q. why do we even study conservation laws!
or say collisions

A. We Study them to understand the Scattering experiments involving Atoms / Atomic particles such as Neutrons / Protons / other elementary particles.
Rutherford Scattering Experiment or Any particle Accelerator of Contemporary world does the same.

Recall

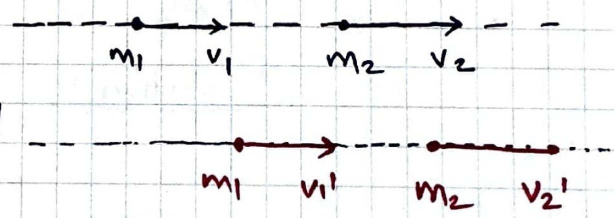
→ Conservation of Momentum Law
in case of Absence of External forces.

→ InElastic Collision where Some of Kinetic energies of colliding particles are converted to heat / light / Sound.

$\boxed{K_i > K_f}$ or $\boxed{K_i - K_f = Q}$

But in Elastic Collision $\boxed{K_i = K_f}$

3.6.1 Collision in One Dimension



For Momentum

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$$

For ~~Set~~ Energy

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_2v_1'^2 + \frac{1}{2}m_2v_2'^2 + Q$$

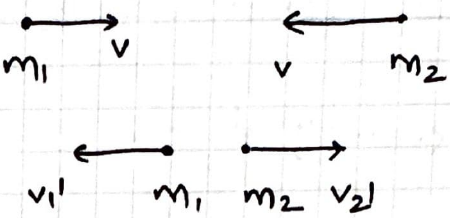
3.6.2 Elastic Collision

Consider 1D motion, where two balls of mass m_1 & m_2 are moving towards each other with speed v.

Mass $m_1 = m$ & $m_2 = 3m$.

We will find the velocities of them after Collision.

Apply conservation of momentum :-



$$P_i = P_f$$

$$\Rightarrow \sum m_i v_i = \sum m_f v_f$$

$$\Rightarrow \cancel{m_1 v} + \cancel{m_2 v} V = m_1$$

$$\Rightarrow m_1 v + m_2(-v) = m_1 v_1' + m_2 v_2' \quad \text{———} \quad (1)$$

Now Applying conservation of energy law :-

$$\frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2 = \frac{1}{2}m_1(v_1')^2 + \frac{1}{2}m_2(v_2')^2 \text{ ———} \quad (II)$$

Solving equation (1) & equation (II) we get

$$v_1' = -2V - 3V_2' \qquad \left\{ \text{used } m_1 = m,\; m_2 = 3m \right\}$$

and thus,

$$4v^2 = 4v^2 + 12VV_2' + 12V_2'^2$$

$$\Rightarrow 0 = 12VV_2' + 12(V_2')^2$$

It will have two solution as it is it quadratic

Hence, Solution 1 $\boxed{V_1' = V \quad V_2' = -V}$

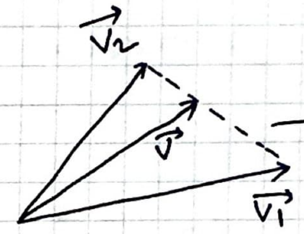
Solution 2 $\boxed{V_1' = -2V \quad V_2' = 0}$

3.6.3 Collision and Center of Mass Coordinates

Because of some symmetry, stating conservations laws of 3D motion, is better explained in center of mass frame of reference rather then Laboratory frame of reference. We will see why!

Consider two particles of mass m_1 and m_2 with velocities $\vec{v_1}$ and $\vec{v_2}$ respectively :-

$$\vec{V_c} = \frac{m_1\vec{v_1} + m_2\vec{v_2}}{m_1+m_2} \quad \text{--- (I)}$$



velocity of center of mass

Now, velocity of mass m_1 with respect to center of mass m_1 is $\vec{V_{1c}} = \vec{V_1} - \vec{V_c}$ --- (II)

Similarly $\vec{V_{2c}} = \vec{V_2} - \vec{V_c}$ --- (III)

Furthermore
$$\vec{V_{1c}} = \vec{V_1} - \left[\frac{m_1\vec{v_1} + m_2\vec{v_2}}{m_1+m_2}\right]$$
$$= \frac{m_2}{m_1+m_2}(\vec{v_1}-\vec{v_2}) \quad \text{--- (IV)}$$

and
$$\vec{V_{2c}} = \frac{-m_1}{m_1+m_2}(\vec{v_1}-\vec{v_2}) \quad \text{--- (V)}$$

Now with velocities with respect to center of mass, we can calculate ~~P~~ Momentum with respect to center of Mass.

$$\boxed{\vec{P_{1c}} = m_1\vec{V_{1c}}} \qquad \boxed{\vec{P_2} = m_2\vec{V_{2c}}}$$

or $$\boxed{\vec{P_{1c}} = \mu\vec{V}} \qquad \boxed{\vec{P_{2c}} = -\mu\vec{V}}$$

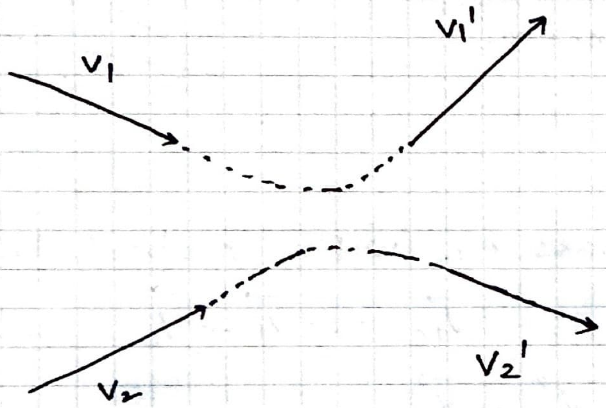
μ = reduced Mass.

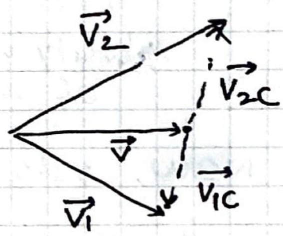
So, Total Momentum of System in C-System is

$$\vec{P_{1c}} + \vec{P_{2c}} = \mu\vec{v} + (-\mu\vec{v}) = 0$$

as it should be as the velocity of System's C.O.M with itself is ofcourse zero.

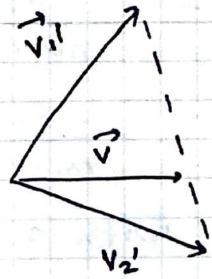
Now let us find Total Momentum in L-System. Once we formulate it, we will relate these with observations in C.O.M frame (i.e C-System).





(a) Trajectories and velocities of two Collision Particles.

(b) Initial velocities in L & C - System



(c) Trajectories and velocities of collision particles after Collision observe that geometric relation in b/w particles as per L-System is quite complicated. But such trajectories in C-system is much simpler to understand.

So, applying conservation of energy in C-system :-

$$\frac{1}{2}m_1(v_{1c})^2 + \frac{1}{2}m_2(v_{2c})^2 = \frac{1}{2}m_1(v_{1c}')^2 + \frac{1}{2}m_2(v_{2c}')^2 \quad (1)$$

and we know Momentum in C-System is zero.

So $m_1\vec{v_{1c}} + m_2\vec{v_{2c}} = 0$

$$\Rightarrow m_1 v_{1c} - m_2 v_{2c} = 0 \quad \text{—— (II)}$$

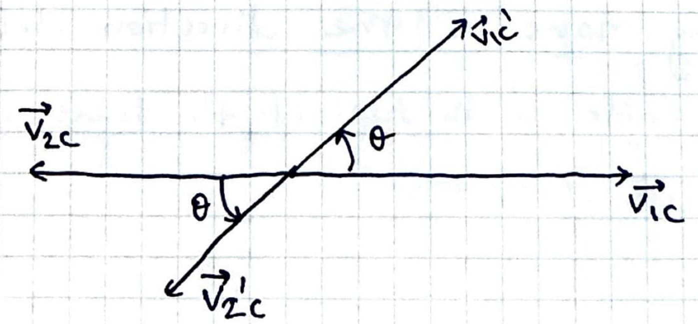
and $m_2 v_{2c}' - m_2 v_{2c}' = 0$ —— (III)

Recall that :- $\vec{P_{1c}} = \mu\vec{v}$, $\vec{P_{2c}} = -\mu\vec{v}$ $\Big\}$ $\vec{P}_{sys.c} = 0$

Momentum of each individual particle is same in Magnitude & opposite in direction.

Means the system is at rest in its own C.O.M

Thus one can observe that particle do get deflected by same angle in their scattering plane.



Thus using eq (II) & eq (III) in eq (I),

we could simplify to write :-

$$m_1 V_{1c} - m_2 V_{2c} = 0$$

$$\Rightarrow \quad V_{2c} = \frac{m_1}{m_c} V_{1c} \qquad \text{—— (IIA)}$$

$$\& \quad V_{2c}' = \frac{m_1}{m_2} V_{1c}' \qquad \text{—— (IIIA)}$$

Now using IIA & IIIA to eliminate V_{2c} & V_{2c}' from eq i we get,

$$\frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right)(V_{1c})^2 = \frac{1}{2}\left(m_1 + \frac{m_1^2}{m_2}\right)(V_{1c}')^2$$

or $\quad V_{1c} = V_{1c}'$

Similarly

$\quad V_{2c} = V_{2c}'$

Hence in elastic Collision the speed of particles in same before & after Collision in C-System.

Key Takeaways

1. Total Momentum of the System in Center of Mass Frame $\rightarrow$ is Zero
2. Momentum of Particles with respect to Center of Mass are equal in Magnitude, but opposite in direction. $\vec{P_{1c}} = \mu \vec{V_c}$ & $\vec{P_{2c}} = -\mu \vec{V_c}$
3. When two particles collide, then after Collision they move in the direction deflected by same angle with respect to their initial direction in C-System.

Collision case when one particle is at rest

Suppose m_2 is at rest

So, we could write

$$\vec{V_c} = \frac{m_1\vec{V_1}}{m_1+m_2} \quad \text{——— (I)}$$

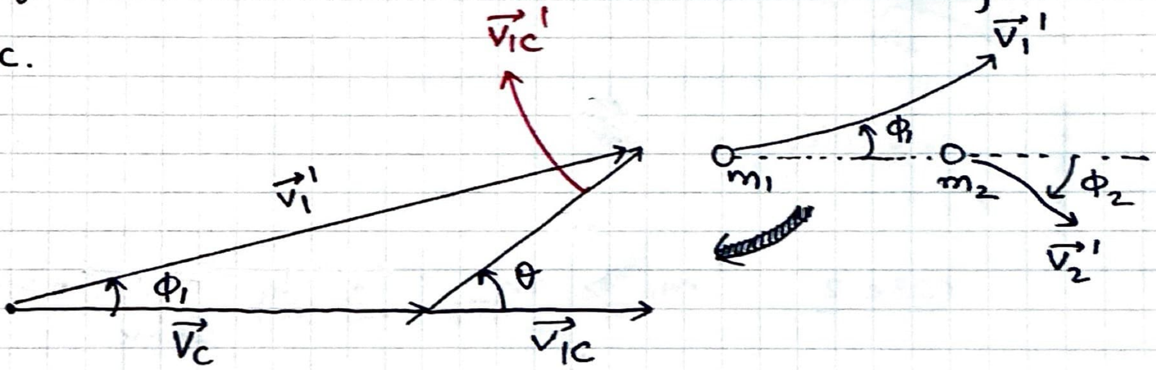
C.O.M's velocity is parallel to $\vec{V_1}$.

$$\text{and } \vec{V_{1c}} = \vec{V_1} - \vec{V_c} \quad \text{——— (II)}$$

$$\vec{V_{2c}} = \vec{V_2} - \vec{V}$$

$$= -\vec{V_c} \quad \text{——— (III)}$$

Now suppose m_1 is scattered by angle θ in C.O.M frame of reference then we could draw following Schematic.



So by using Trignometry, we could write:-

$$\tan\phi_1 = \frac{|\vec{V_{1c}}|\cdot\sin\theta}{|\vec{V_c}| + |\vec{V_{1c}}'|\cdot\cos\theta}$$

Now as Scattering is Elastic, Hence $V_{1c}' = V_{1c}$

we could write:-

$$\tan(\phi_1) = \frac{V_{1c}\cdot\sin\theta}{V_c + V_{1c}\cos\theta}$$

$$= \frac{\sin\theta}{V_c/V_{1c} + \cos\theta} \quad \left\{ \begin{array}{l}\text{Divided Numerator} \\ \text{and denominator by } V_{1c}\end{array} \right\}$$

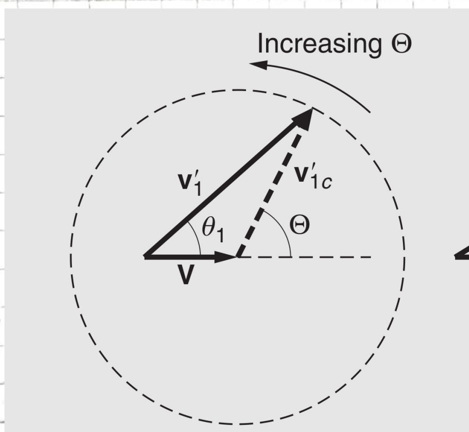
$$\Rightarrow \tan\phi_1 = \frac{\sin\theta}{(m_1/m_2) + \cos\theta} \quad \left\{ \begin{array}{l}\text{we used eq(I), (II), (III)} \\ \text{to find } V_c/V_{1c}\end{array} \right\}$$

So for elastic collision, when a particle is at rest and another is in Motion.

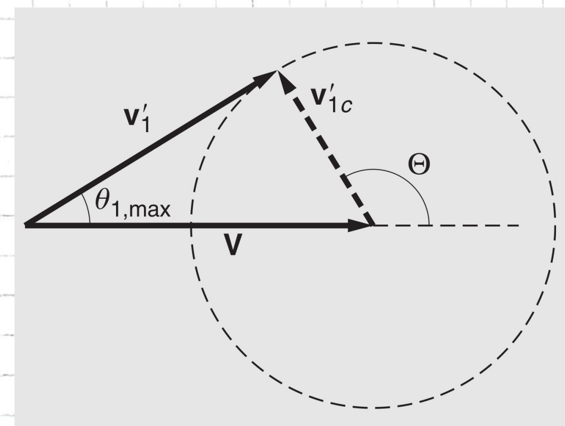
So $\tan\phi_1 = \dfrac{\sin\theta}{(m_1/m_2) + \cos\theta}$

Now scattering angle θ depends upon the nature of scattering force.

<u>case 1</u> $m_1 < m_2 \longrightarrow \phi_1$ is unrestricted.



<u>case 2</u> $m_1 > m_2 \longrightarrow \phi_1$ has maximum value given by



$$\sin\phi_{1max} = \frac{V_{1c}}{V_c} = \frac{m_2}{m_1}$$

<u>case 3</u> $m_1 \gg m_2 \longrightarrow \phi_1$ approaches zero

Thus that's why light object at rest can not appreciably deflects the incoming particle.

~~3.6.4 Relation of Recoil Angle in L-System and scatter~~

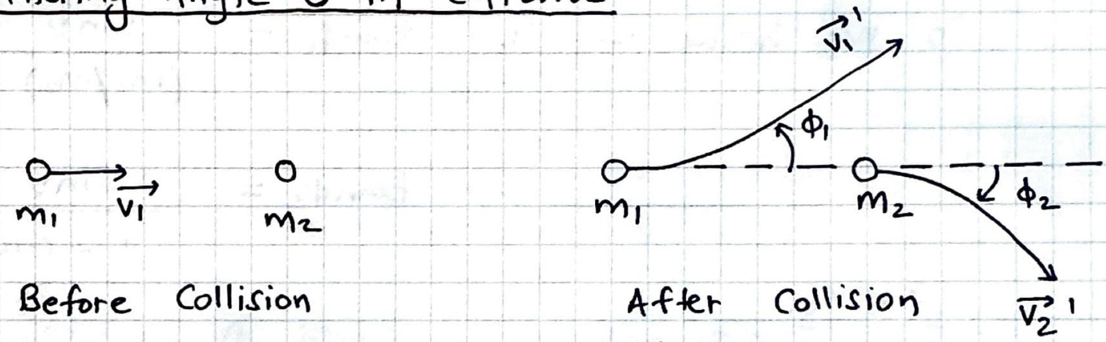
S-PYQ

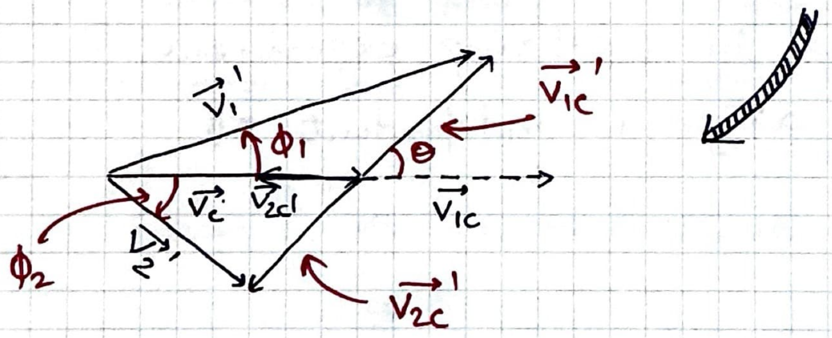
Establish that is object is at rest then incoming particle can not be deflected

~~3.6.4 Relation of Recoil Angle~~ ϕ_2 ~~&~~ in L frame &

O

3.6.4 Relation of Recoil Angle (ϕ_2) in L frame and Scattering Angle θ in C frame





from geometry we could write that :-

$$\tan\phi_2 = \frac{V_{2c}\sin\theta}{V_c - V_{2c}\cos\theta}$$

$$= \frac{\sin\theta}{|(V_c/V_{2c})| - \cos\theta}$$

$$= \frac{\sin\theta}{1 - \cos\theta}$$

$$\Rightarrow \tan\phi_2 = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\sin^2(\theta/2)} = \cot\left(\frac{\theta}{2}\right)$$

So, $\tan \phi_2 = \cot\left(\frac{\theta}{2}\right)$

$$= \tan\left(\frac{\pi}{2} - \frac{\theta}{2}\right)$$

or $\phi_2 = \frac{\pi}{2} - \frac{\theta}{2}$

or $2\phi_2 = \pi - \theta$

Thus one can say angle of recoil of the target particle is independent of masses.

So we have :- I. $\tan\phi_1 = \frac{\sin\theta}{(m_1/m_2) + \cos\theta}$

II. $\tan\phi_2 = \frac{\sin\theta}{1 - \cos\theta}$

S-PYQ

Prove that in scattering experiment the recoil angle

3.6.6 Maximum Angle of Scattering in L Frame

Maximum Angle would be given by the derivative of

$$\tan\phi_1 = \frac{\sin\theta}{(m_1/m_2) + \cos\theta} \quad \text{with respect to } \theta. \qquad \text{— (I)}$$

Hence
$$\frac{d}{d\theta}(\tan\phi_1) = \frac{d}{d\theta}\left[\sin\theta\left(\cos\theta + \frac{m_1}{m_2}\right)^{-1}\right]$$
$$= \cos\theta\left[\left(\cos\theta + m_1/m_2\right)^{-1}\right]$$
$$+$$
$$\sin\theta\left[-\left(\cos\theta + \frac{m_1}{m_2}\right)^{-2}\right]\left[-\sin\theta\right]$$

or
$$\frac{d}{d\theta}(\tan\phi_1) = \frac{1 + (m_1/m_2)\cos\theta}{(\cos\theta + m_1/m_2)^2}$$

~~for maximum of $\tan\phi_1$, denominator of above should be minimum. Hence $\cos\theta + m_1/m_2$~~

for maxima of $\tan\phi_1$, its derivative should be equal to zero & hence double derivative should be negative.

Thus
$$1 + (m_1/m_2)\cos\theta = 0$$
$$\cos\theta = -m_2/m_1$$
$$\Rightarrow \quad 1 - \sin^2\theta = (m_2/m_1)^2$$
$$\Rightarrow \quad \sin^2\theta = 1 - (m_2/m_1)^2$$
$$\Rightarrow \quad \sin\theta = \sqrt{1 - (m_2/m_1)^2} \qquad \text{— (II)}$$

using eq (II) in eq (I), we get :-
$$(\tan\phi_1)_{max} = \frac{\left[1 - (m_2/m_1)^2\right]^{+1/2}}{m_1/m_2 + (-m_2/m_1)}$$

or
$$(\phi_1)_{max} = \tan^{-1}\left[\frac{m_2}{m_1^2 - m_2^2}\right]$$

3.67 Energies of Colliding particle (m_2 at rest)

Kinetic Energy relationship is given by $T_0 = T_0' + T_c$.
where (I) T_0 is total kinetic energy before collision is Laboratory frame. (II) T_0' is total kinetic energy before Collision in C.O.M frame (III) T_c is Kinetic energy of C.O.M itself w.r. to Laboratory frame.

Hence $\quad T_0 = \frac{1}{2} m_1 v_1^2 \quad$ as $\quad V_2 = 0$

$$T_0' = \frac{1}{2} m_1 (v_{1c})^2 + \frac{1}{2} m_2 (v_{2c})^2$$

$$T_c = \frac{1}{2} (m_1 + m_2) V_c^2$$

Additionally, one can establish $\frac{T_0}{T_0'} = 1 + m_1/m_2$

Note that $\quad \vec{V_c} = \frac{m_1 \vec{V_1}}{m_1 + m_2}$

$$\vec{V_{1C}} = \vec{V_1} - \vec{V_c} = \vec{V_1} - \frac{m_1 \vec{V_1}}{m_1 + m_2} = \frac{m_2 \vec{V_1}}{m_1 + m_2}$$

$$\vec{V_{2C}} = \vec{V_2} - \vec{V_c} = -\vec{V_c} = -\frac{m_1 \vec{V_1}}{m_1 + m_2}$$

Also $\quad T_{1c} = \frac{1}{2} m_1 (v_{1c})^2$

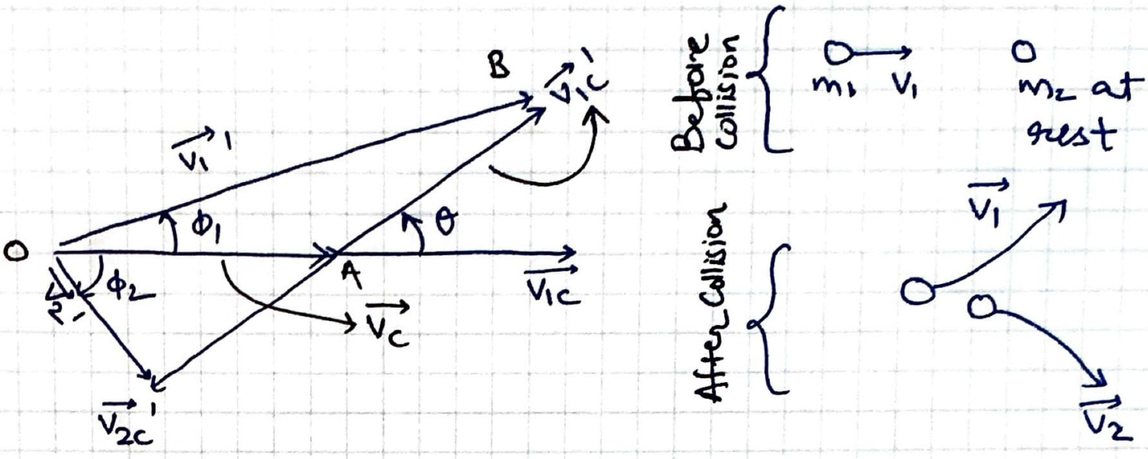
$$= \frac{1}{2} m_1 \left(\frac{m_2 V_1}{m_1 + m_2}\right)^2 = \left(\frac{m_2}{m_1 + m_2}\right)^2 T_0$$

Similarly

$$T_{2c} = \frac{m_1 m_2}{(m_1 + m_2)^2} \cdot T_0$$

finally,
$$\boxed{\frac{T_{1c}}{T_{2c}} = \frac{m_2}{m_1}}$$

3.6 8 Kinetic energies in elastic Collision in L frame in term of Scattering angle θ in C frame



Laboratory frame

Kinetic energy before Collision is $T_0 = \frac{1}{2} m_1 V_1^2$

" " a " for m_1 is : $T_1 = \frac{1}{2} m_1 (V_1')^2$

" " after " for m_2 is : $T_2 = \frac{1}{2} m_2 (V_2')^2$

So, $\frac{T_1}{T_0} = \frac{V_1'^2}{V_1^2}$ —— (I)

V_1' → velocity in L frame after collision

V_1 → velocity in L frame before collision

Now from the sketch, we could write:-

from ΔOAB : $(V_{1c}')^2 = (V_1')^2 + (V_c)^2 - 2 \cdot V_1' \cdot V_c \cdot \cos\phi_1$

or $(V_1')^2 = (V_{1c}')^2 - (V_c)^2 + 2 \cdot V_1' \cdot V_c \cdot \cos\phi_1$ —— (II)

Now substituting eq (II) in eq (I), we get:-

$$\frac{T_1}{T_0} = \frac{(V_{1c}')^2}{V_1^2} - \frac{V_c^2}{V_1^2} + \frac{2 \cdot V_1' \cdot V_c \cdot \cos\phi_1}{V_1^2}$$

$$\Rightarrow \frac{T_1}{T_0} = 1 - \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) \quad \text{—— (III)}$$

Similarly, $$\frac{T_2}{T_0} = \frac{2m_1m_2}{(m_1+m_2)^2}(1-\cos\theta) \quad \text{—— (IV)}$$

Hence eq (III) and eq (IV) relates K.E to Scattering angle.