UPSC PHYSICS PYQ SOLUTION
Mechanics - Part 2

Contents

11 A rod of length l has non-uniform linear mass density (mass per unit length) λ, which varies as $\lambda = \lambda_0\left(\frac{x}{L}\right)$, where λ_0 is a constant and x is the distance from the end marked '0'. Find the centre of mass of the rod.

Introduction: We are given a rod of total length l with a variable linear mass density given by $\lambda(x) = \lambda_0\left(\frac{x}{L}\right)$, where x is measured from one end (marked '0') of the rod. The goal is to find the location of the center of mass (COM) of the rod.

Here, λ_0 and L are constants. The function $\lambda(x)$ increases linearly with x, indicating the rod is heavier towards the end with larger x. We assume the mass is distributed along a straight line and lies along the x-axis from $x = 0$ to $x = l$.

Solution:

1. Total mass of the rod:

The mass element at position x is:

$$dm = \lambda(x)\,dx = \lambda_0\left(\frac{x}{L}\right)\,dx$$

Integrating from 0 to l:

$$M = \int_0^l dm = \int_0^l \lambda_0\left(\frac{x}{L}\right)dx = \frac{\lambda_0}{L}\int_0^l x\,dx = \frac{\lambda_0}{L}\cdot\frac{l^2}{2}$$

So,

$$M = \frac{\lambda_0 l^2}{2L}$$

2. Center of mass:

The position of the center of mass x_{cm} is given by:

$$x_{\text{cm}} = \frac{1}{M}\int_0^l x\,dm = \frac{1}{M}\int_0^l x\,\lambda(x)\,dx = \frac{1}{M}\int_0^l x\,\lambda_0\left(\frac{x}{L}\right)dx$$

Simplify the integrand:

$$x_{\text{cm}} = \frac{1}{M}\cdot\frac{\lambda_0}{L}\int_0^l x^2\,dx = \frac{\lambda_0}{ML}\cdot\frac{l^3}{3}$$

Now substitute $M = \frac{\lambda_0 l^2}{2L}$:

$$x_{\text{cm}} = \frac{\lambda_0 l^3}{3L}\cdot\frac{2L}{\lambda_0 l^2} = \frac{2l}{3}$$

Conclusion:

The center of mass of the rod lies at a distance $\boxed{\frac{2l}{3}}$ from the end marked '0'. This is closer to the denser end of the rod, as expected due to the increasing mass density with x.

12 (i) What is a central force? Give two examples of the central force.

Introduction: In classical mechanics, forces acting on particles can be categorized based on their nature and point of application. A particular class of forces known as *central forces* plays a fundamental role in orbital mechanics and the study of conservative systems.

Solution:

A **central force** is defined as a force that:

1. Always acts along the line joining the particle and a fixed point (the center), and 2. Has a magnitude that depends only on the distance between the particle and the center.

Mathematically, if $\vec{r}$ is the position vector of a particle relative to a fixed point (origin), a central force $\vec{F}$ can be expressed as:

$$\vec{F}(\vec{r}) = f(r)\,\hat{r}$$

where:

1. $r = |\vec{r}|$ is the radial distance from the center,
2. $\hat{r} = \frac{\vec{r}}{r}$ is the unit vector in the direction of $\vec{r}$,
3. $f(r)$ is a scalar function depending only on r.

Key properties of central forces:

1. They are **radial** (directed along $\vec{r}$).
2. They are **conservative**, meaning the work done is path-independent and a potential function $V(r)$ exists.
3. They conserve **angular momentum**, which leads to planar motion of the particle.

Examples of central forces:

1. **Gravitational Force**:

$$\vec{F}_g = -\frac{GMm}{r^2}\,\hat{r}$$

where G is the gravitational constant, M and m are the interacting masses, and r is the distance between them. This force is attractive and follows an inverse square law.

2. **Electrostatic (Coulomb) Force**:

$$\vec{F}_e = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q_1 q_2}{r^2}\,\hat{r}$$

where q_1 and q_2 are the charges, ε_0 is the vacuum permittivity, and r is the separation. This force can be attractive or repulsive depending on the sign of charges.

Conclusion:

A central force is a radial, distance-dependent force acting between a particle and a fixed center. Gravitational and electrostatic forces are classic examples, both following the inverse-square law and playing central roles in planetary and atomic dynamics.

(ii) Show that the angular momentum (L) of the particle in a central force field is a constant of motion.

Introduction: In classical mechanics, a central force is a force that is always directed along the line joining a particle and a fixed point (the center), and whose magnitude depends only on the distance from the center. The angular momentum $\vec{L}$ of a particle is defined as $\vec{L} = \vec{r} \times \vec{p}$, where $\vec{r}$ is the position vector and $\vec{p}$ is the linear momentum $\vec{p} = m\vec{v}$. We are asked to show that $\vec{L}$ is conserved, i.e., $\dfrac{d\vec{L}}{dt} = 0$, for a particle under the influence of a central force.

Solution:

Let a particle of mass m move under the influence of a central force $\vec{F}$, directed along the radial vector $\vec{r}$ from a fixed point (the origin). Then,

$$\vec{F} = f(r)\,\hat{r}$$

where $f(r)$ is some scalar function of the radial distance $r = |\vec{r}|$, and $\hat{r} = \vec{r}/r$ is the radial unit vector.

The angular momentum of the particle is defined as:

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}$$

To determine whether $\vec{L}$ is conserved, we take its time derivative:

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v}) = m\frac{d}{dt}(\vec{r} \times \vec{v})$$

Using the product rule for the time derivative of a cross product:

$$\frac{d\vec{L}}{dt} = m\left(\frac{d\vec{r}}{dt} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right)$$

Now, $\frac{d\vec{r}}{dt} = \vec{v}$, and $\vec{v} \times \vec{v} = \vec{0}$, so:

$$\frac{d\vec{L}}{dt} = m\left(\vec{v} \times \vec{v} + \vec{r} \times \frac{d\vec{v}}{dt}\right) = m\vec{r} \times \frac{d\vec{v}}{dt}$$

Using Newton's second law, $\frac{d\vec{v}}{dt} = \frac{\vec{F}}{m}$:

$$\frac{d\vec{L}}{dt} = m\vec{r} \times \left(\frac{\vec{F}}{m}\right) = \vec{r} \times \vec{F}$$

Now, since $\vec{F}$ is a central force, it is directed along $\vec{r}$:

$$\vec{F} = f(r)\,\hat{r} = f(r)\frac{\vec{r}}{r}$$

Then,

$$\vec{r} \times \vec{F} = \vec{r} \times \left(f(r)\frac{\vec{r}}{r}\right) = \frac{f(r)}{r}(\vec{r} \times \vec{r}) = \vec{0}$$

Hence,

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = \vec{0}$$

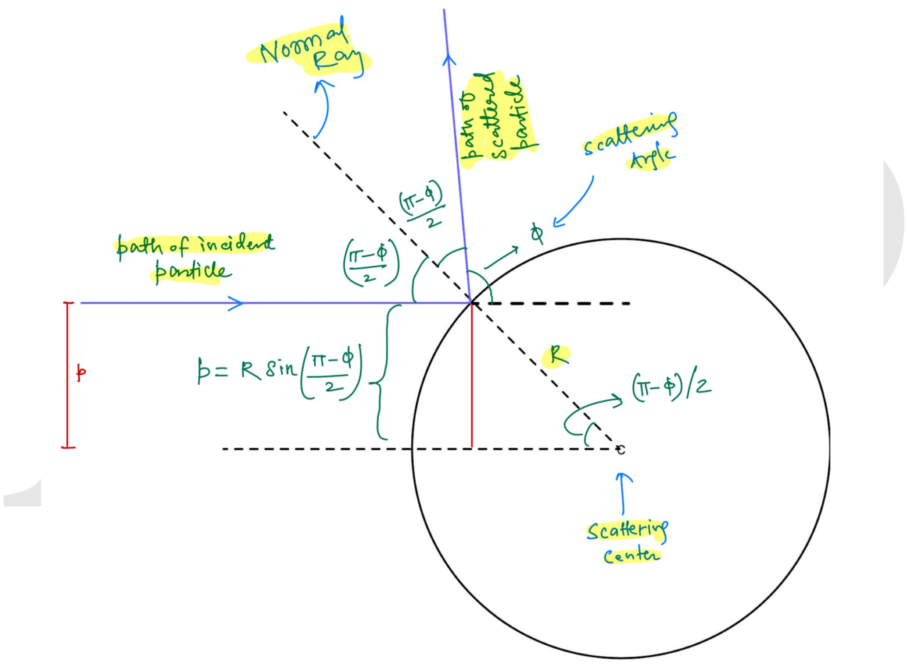
Conclusion: The angular momentum $\vec{L}$ of a particle moving under a central force is conserved, i.e., $\frac{d\vec{L}}{dt} = 0$. This result reflects the rotational symmetry of the system and is a consequence of Noether's theorem, which relates conservation laws to symmetries of the system.

13 Show that the differential scattering cross-section for elastic scattering of a point particle from an infinitely massive sphere of radius R is $\frac{R^2}{4}$. What is the inference of this result?

Introduction:

We are asked to derive the differential scattering cross-section for the elastic scattering of a point particle from an infinitely massive, rigid sphere of radius R. The process is purely geometric and involves specular reflection at the point of contact with the sphere. Let the scattering angle be denoted by ϕ, defined as the angle between the initial and final velocity vectors of the particle. The goal is to show that the differential scattering cross-section is constant and equals $\frac{R^2}{4}$, and to interpret the physical meaning of this result.

Solution:



From the geometry of the scattering event (refer to the earlier setup), the impact parameter p is related to the scattering angle ϕ by:

$$p = R\cos\left(\frac{\phi}{2}\right)$$

Differentiating this expression with respect to ϕ gives:

$$\frac{dp}{d\phi} = -\frac{R}{2}\sin\left(\frac{\phi}{2}\right)$$

The differential scattering cross-section is given by:

$$\frac{d\sigma}{d\Omega} = \frac{p}{\sin\phi}\left|\frac{dp}{d\phi}\right|$$

Substituting the expressions for p and $\frac{dp}{d\phi}$:

$$\begin{aligned}\frac{d\sigma}{d\Omega} &= \frac{R\cos\left(\frac{\phi}{2}\right)}{\sin\phi}\cdot\left(\frac{R}{2}\sin\left(\frac{\phi}{2}\right)\right)\\ &= \frac{R^2}{2}\cdot\frac{\cos\left(\frac{\phi}{2}\right)\sin\left(\frac{\phi}{2}\right)}{\sin\phi}\end{aligned}$$

Now use the trigonometric identity:

$$\sin\phi = 2\sin\left(\frac{\phi}{2}\right)\cos\left(\frac{\phi}{2}\right)$$

Substituting this into the expression yields:

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{2}\cdot\frac{1}{2} = \frac{R^2}{4}$$

Conclusion:

The differential scattering cross-section for the elastic scattering of a point particle from an infinitely massive, rigid sphere of radius R is:

$$\frac{d\sigma}{d\Omega} = \frac{R^2}{4}$$

This result implies that the scattering is isotropic, with equal probability into all solid angles, and the cross-section is independent of the scattering angle ϕ. Physically, it reflects the purely geometric nature of hard-sphere scattering, where each incident direction corresponds uniquely to a point on the sphere's surface. This result is characteristic of elastic, specular reflection off a hard, spherical obstacle.

14 A rocket starts vertically upwards with speed v_0. Then define its speed v at a height h in terms of h, R (radius of Earth) and g (acceleration due to gravity on Earth's surface). Also calculate the maximum height attained by a rocket fired with a speed of 90% of the escape velocity.

Introduction: We are given a rocket that starts vertically upward from the Earth's surface with an initial speed v_0. The goal is to express the speed v of the rocket as a function of height h above Earth's surface, using the gravitational constant g at the surface and the Earth's radius R. In the second part, we are to determine the maximum height attained by a rocket launched with a speed equal to 90% of the escape velocity from Earth.

Assumptions:

(i) Gravity decreases with altitude as an inverse-square law.

(ii) Air resistance is neglected.

(iii) The Earth is a uniform sphere with mass M and radius R.

Solution:

The total mechanical energy E of the rocket at any point is the sum of kinetic energy and gravitational potential energy. Using conservation of mechanical energy:

At the surface of the Earth:

$$E_{\text{initial}} = \frac{1}{2}mv_0^2 - \frac{GMm}{R}$$

At a height h above the surface (distance from center is $R + h$):

$$E_{\text{final}} = \frac{1}{2}mv^2 - \frac{GMm}{R+h}$$

Equating the two energies:

$$\frac{1}{2}mv^2 - \frac{GMm}{R+h} = \frac{1}{2}mv_0^2 - \frac{GMm}{R}$$

Cancel m and rearrange:

$$\frac{1}{2}v^2 = \frac{1}{2}v_0^2 + GM\left(\frac{1}{R+h} - \frac{1}{R}\right)$$

Now relate GM to g using $g = \dfrac{GM}{R^2}$, so $GM = gR^2$. Substituting:

$$\frac{1}{2}v^2 = \frac{1}{2}v_0^2 + gR^2\left(\frac{1}{R+h} - \frac{1}{R}\right)$$

Simplify the right-hand side:

$$\frac{1}{2}v^2 = \frac{1}{2}v_0^2 - gR^2\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

Therefore, the speed v at height h is:

$$v^2 = v_0^2 - 2gR^2 \left(\frac{1}{R} - \frac{1}{R+h}\right)$$

Maximum Height:

At the maximum height, the rocket's speed becomes zero, i.e., $v = 0$. Using the equation above:

$$0 = v_0^2 - 2gR^2 \left(\frac{1}{R} - \frac{1}{R+h_{\text{max}}}\right)$$

Rearranging:

$$v_0^2 = 2gR^2 \left(\frac{1}{R} - \frac{1}{R+h_{\text{max}}}\right)$$

Let us now express v_0 in terms of the escape velocity v_{esc}:

$$v_{\text{esc}} = \sqrt{2gR} \quad \Rightarrow \quad v_0 = 0.9v_{\text{esc}} = 0.9\sqrt{2gR}$$

So:

$$v_0^2 = 0.81 \cdot 2gR = 1.62gR$$

Now plug into the equation:

$$1.62gR = 2gR^2 \left(\frac{1}{R} - \frac{1}{R+h_{\text{max}}}\right)$$

Simplify:

$$1.62 = 2R \left(\frac{1}{R} - \frac{1}{R+h_{\text{max}}}\right)$$

Divide both sides by 2:

$$0.81 = R \left(\frac{1}{R} - \frac{1}{R+h_{\text{max}}}\right)$$

Simplify left-hand side:

$$0.81 = 1 - \frac{R}{R+h_{\text{max}}}$$

Therefore:

$$\frac{R}{R+h_{\text{max}}} = 0.19 \quad \Rightarrow \quad R + h_{\text{max}} = \frac{R}{0.19}$$

Then:

$$h_{\text{max}} = \frac{R}{0.19} - R = R\left(\frac{1}{0.19} - 1\right) = R\left(\frac{1-0.19}{0.19}\right) = R\left(\frac{0.81}{0.19}\right)$$

Calculate:

$$h_{\text{max}} \approx 4.263R$$

Conclusion: The speed of the rocket at a height h is given by

$$v^2 = v_0^2 - 2gR^2\left(\frac{1}{R} - \frac{1}{R+h}\right)$$

If the rocket is launched with 90% of escape velocity, the maximum height it attains is approximately $\boxed{4.263\,R}$ above the Earth's surface.

15 A particle moving in a central force field describes the path $r = ke^{\alpha\theta}$, where k and α are constants. If the mass of the particle is m, find the law of force.

Introduction: We are given that a particle of mass m moves under a central force and its trajectory is given in polar coordinates as $r = ke^{\alpha\theta}$. The task is to determine the corresponding central force law $F(r)$, i.e., to find the force as a function of the radial distance r from the origin. Since the force is central, it depends only on r, and we can use the standard results from central force motion and orbit theory.

Solution:

We use the general orbit equation in central force motion:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{l^2u^2}F\left(\frac{1}{u}\right)$$

where $u = \frac{1}{r}$ and l is the angular momentum per unit mass.

Given the path is $r = ke^{\alpha\theta}$, take the reciprocal:

$$u = \frac{1}{r} = \frac{1}{k}e^{-\alpha\theta}$$

Differentiate with respect to θ:

$$\frac{du}{d\theta} = -\frac{\alpha}{k}e^{-\alpha\theta} = -\alpha u$$
$$\frac{d^2u}{d\theta^2} = \alpha^2 u$$

Now plug into the orbit equation:

$$\frac{d^2u}{d\theta^2} + u = \alpha^2 u + u = (1+\alpha^2)u$$

Thus:

$$(1+\alpha^2)u = -\frac{m}{l^2u^2}F\left(\frac{1}{u}\right)$$

Solve for $F(r)$ by expressing everything in terms of r:

$$u = \frac{1}{r}$$
$$u^3 = \frac{1}{r^3}$$
$$F(r) = -\frac{l^2}{m}(1+\alpha^2)u^3 = -\frac{l^2}{m}(1+\alpha^2)\frac{1}{r^3}$$

So the force law is:

$$F(r) = -\frac{l^2}{m}(1+\alpha^2)\frac{1}{r^3}$$

This shows that the central force varies inversely with the cube of the distance from the center.

Conclusion: The particle moves under a central force that follows the inverse-cube law:

$$\boxed{F(r) = -\frac{l^2}{m}(1+\alpha^2)\frac{1}{r^3}}$$

This is a repulsive or attractive inverse-cube force depending on the sign convention. Such a force is distinct from the more familiar inverse-square laws (e.g., gravitational or electrostatic forces), and it leads to non-closed, spiral or exponential trajectories such as the given logarithmic spiral.

16 Calculate the mass and momentum of a proton of rest mass 1.67×10^{-27} kg moving with a velocity of $0.8c$, where c is the velocity of light. If it collides and sticks to a stationary nucleus of mass 5.70×10^{-26} kg, find the velocity of the resultant particle.

Introduction: We are given a proton of rest mass $m_0 = 1.67 \times 10^{-27}$ kg moving at a speed $v = 0.8c$, where $c = 3.00 \times 10^8$ m/s. The proton collides with and sticks to a stationary nucleus of mass $M = 5.70 \times 10^{-26}$ kg. The tasks are:

(i) Calculate the relativistic mass and momentum of the proton.

(ii) Determine the velocity of the resulting composite particle after the perfectly inelastic collision.

We will use relativistic dynamics since the proton's speed is a significant fraction of the speed of light.

Solution:

(i) Relativistic mass and momentum of the proton

The relativistic mass m is given by:

$$m = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Substitute $v = 0.8c$:

$$\begin{aligned}
&\frac{v^2}{c^2} = (0.8)^2 = 0.64 \\
&\Rightarrow m = \frac{1.67 \times 10^{-27}}{\sqrt{1 - 0.64}} = \frac{1.67 \times 10^{-27}}{\sqrt{0.36}} = \frac{1.67 \times 10^{-27}}{0.6} \\
&\Rightarrow m \approx 2.783 \times 10^{-27}\ \text{kg}
\end{aligned}$$

The relativistic momentum p is:

$$p = mv = 2.783 \times 10^{-27} \times 0.8 \times 3.00 \times 10^8$$

$$\begin{aligned}
p &= 2.783 \times 0.8 \times 3.00 \times 10^{-27+8} \\
&= 6.6792 \times 10^{-19}\ \text{kg} \cdot \text{m/s}
\end{aligned}$$

(ii) Velocity of the resultant particle

Let V be the final velocity of the composite particle after the perfectly inelastic collision.

Using conservation of linear momentum:

$$\text{Total initial momentum} = \text{Total final momentum}$$

Before collision:

$$p_{\text{initial}} = p_{\text{proton}} + p_{\text{nucleus}} = 6.6792 \times 10^{-19} + 0 = 6.6792 \times 10^{-19}\,\text{kg} \cdot \text{m/s}$$

After collision, the total mass is:

$$m_{\text{total}} = m + M = 2.783 \times 10^{-27} + 5.70 \times 10^{-26} = 5.9783 \times 10^{-26}\,\text{kg}$$

Let the final velocity be V, then:

$$p_{\text{final}} = m_{\text{total}} \cdot V$$

Equating:

$$6.6792 \times 10^{-19} = 5.9783 \times 10^{-26} \cdot V \quad \Rightarrow \quad V = \frac{6.6792 \times 10^{-19}}{5.9783 \times 10^{-26}}$$

Compute:

$$V \approx 1.1175 \times 10^{7}\,\text{m/s}$$

Conclusion:

(i) The relativistic mass of the proton is $\boxed{2.783 \times 10^{-27}\,\text{kg}}$.

(ii) The relativistic momentum of the proton is $\boxed{6.68 \times 10^{-19}\,\text{kg} \cdot \text{m/s}}$.

(iii) The velocity of the resulting composite particle after the inelastic collision is $\boxed{1.12 \times 10^{7}\,\text{m/s}}$.

17 Show that the mean kinetic and potential energies of non-dissipative simple harmonic vibrating systems are equal.

Introduction: We consider a simple harmonic oscillator (SHO), such as a mass-spring system, that is undamped (non-dissipative). The displacement of the mass from equilibrium follows the equation $x(t) = A\cos(\omega t + \phi)$, where A is the amplitude, ω is the angular frequency, and ϕ is the phase constant. We aim to show that, over one complete oscillation cycle, the time-averaged kinetic energy equals the time-averaged potential energy.

Solution:

Let the mass of the oscillator be m and the spring constant be k. The angular frequency of the system is:

$$\omega = \sqrt{\frac{k}{m}}$$

Kinetic Energy: The velocity is given by:

$$v(t) = \frac{dx}{dt} = -A\omega\sin(\omega t + \phi)$$

So the instantaneous kinetic energy is:

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}mA^2\omega^2\sin^2(\omega t + \phi)$$

Potential Energy: The restoring force is $F = -kx$, so the potential energy stored in the spring is:

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kA^2\cos^2(\omega t + \phi)$$

But since $k = m\omega^2$, we can also write:

$$U(t) = \frac{1}{2}m\omega^2A^2\cos^2(\omega t + \phi)$$

Time-Averaged Energies:

The time average of a function $f(t)$ over one period $T = \frac{2\pi}{\omega}$ is:

$$\langle f\rangle = \frac{1}{T}\int_0^T f(t)\,dt$$

We will now compute the average of $\sin^2(\omega t + \phi)$ explicitly:

$$\frac{1}{T}\int_0^T \sin^2(\omega t + \phi)\,dt = \frac{1}{T}\int_0^T \frac{1-\cos(2\omega t + 2\phi)}{2}\,dt = \frac{1}{2} - \frac{1}{2T}\int_0^T \cos(2\omega t + 2\phi)\,dt$$

Since $\cos(2\omega t + 2\phi)$ completes an integer number of cycles over $[0, T]$, its integral over one period is zero:

$$\Rightarrow \frac{1}{T}\int_0^T \sin^2(\omega t + \phi)\,dt = \frac{1}{2}$$

Similarly,

$$\frac{1}{T}\int_0^T \cos^2(\omega t+\phi)\,dt = \frac{1}{2}$$

Average Kinetic Energy:

$$\langle K\rangle = \frac{1}{T}\int_0^T \frac{1}{2}mA^2\omega^2\sin^2(\omega t+\phi)\,dt = \frac{1}{2}mA^2\omega^2\cdot\frac{1}{2} = \frac{1}{4}mA^2\omega^2$$

Average Potential Energy:

$$\langle U\rangle = \frac{1}{T}\int_0^T \frac{1}{2}m\omega^2A^2\cos^2(\omega t+\phi)\,dt = \frac{1}{2}mA^2\omega^2\cdot\frac{1}{2} = \frac{1}{4}mA^2\omega^2$$

Conclusion: The time-averaged kinetic energy and potential energy of a non-dissipative simple harmonic oscillator are both equal to

$$\boxed{\frac{1}{4}mA^2\omega^2}$$

Hence, the mean kinetic and potential energies are equal in simple harmonic motion. This reflects the energy symmetry of SHM, where energy oscillates between kinetic and potential forms.

18 Show that for very small velocity, the equation for kinetic energy, $K = \Delta mc^2$ becomes $K = \frac{1}{2}m_0v^2$, where notations have their usual meanings.

Introduction: In special relativity, the total energy of a particle of rest mass m_0 moving with velocity v is given by $E = \gamma m_0c^2$, where $\gamma = \dfrac{1}{\sqrt{1 - v^2/c^2}}$ is the Lorentz factor. The kinetic energy is defined as the excess energy above the rest energy:

$$K = E - m_0c^2 = (\gamma - 1)m_0c^2$$

We aim to show that in the limit of very small velocity ($v \ll c$), this expression reduces to the classical kinetic energy formula $K = \frac{1}{2}m_0v^2$.

Solution:

Let us expand γ in a binomial series for $v \ll c$.

Recall:

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-1/2}$$

Using the binomial expansion for $(1 + x)^n$ where $|x| \ll 1$:

$$(1 + x)^n \approx 1 + nx + \frac{n(n-1)}{2}x^2 + \cdots$$

Let $x = -\dfrac{v^2}{c^2}$ and $n = -\dfrac{1}{2}$:

$$\begin{aligned}\gamma &= \left(1 - \frac{v^2}{c^2}\right)^{-1/2} \\ &\approx 1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\left(\frac{v^2}{c^2}\right)^2 + \cdots\end{aligned}$$

Then:

$$\begin{aligned}K &= (\gamma - 1)m_0c^2 \\ &= \left(\frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4} + \cdots\right)m_0c^2 \\ &= \frac{1}{2}m_0v^2 + \frac{3}{8}m_0\frac{v^4}{c^2} + \cdots\end{aligned}$$

Since $v \ll c$, the term $\frac{v^4}{c^2}$ becomes extremely small compared to v^2, and higher-order terms are even smaller. Thus, these can be neglected in the low-velocity approximation.

Therefore:

$$K \approx \frac{1}{2}m_0v^2$$

Conclusion: In the limit of small velocity ($v \ll c$), the relativistic expression for kinetic energy,

$$K = (\gamma - 1)m_0c^2,$$

reduces to the classical expression

$$\boxed{K = \frac{1}{2}m_0v^2}.$$

This demonstrates the consistency of relativistic mechanics with Newtonian mechanics in the low-velocity regime.

19 A particle P of mass m_1 collides with another particle Q of mass m_2 at rest. The particles P and Q travel at angles θ and ϕ, respectively, with respect to the initial direction of P. Derive the expression for the maximum value of θ.

Introduction: A particle P of mass m_1 initially moves with velocity u and collides elastically with a stationary particle Q of mass m_2. After the collision, P and Q move with velocities v_1 and v_2 at angles θ and ϕ with respect to the initial direction of P. The aim is to derive the expression for the **maximum possible value** of the deflection angle θ of particle P.

Assumptions:

(i) The collision is elastic.

(ii) Momentum and kinetic energy are conserved.

(iii) All motion is confined to a plane.

Solution:

Step 1: Conservation Laws

Let the initial velocity of P be u, and the final velocities be v_1 and v_2 for P and Q, respectively.

Conservation of momentum: Along the initial direction (x-axis):

$$m_1 u = m_1 v_1 \cos\theta + m_2 v_2 \cos\phi \quad (1)$$

Perpendicular to initial direction (y-axis):

$$0 = m_1 v_1 \sin\theta - m_2 v_2 \sin\phi \quad (2)$$

Conservation of kinetic energy:

$$\frac{1}{2}m_1 u^2 = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 \quad (3)$$

Step 2: Eliminate v_2 and ϕ

From equation (2):

$$m_1 v_1 \sin\theta = m_2 v_2 \sin\phi \quad \Rightarrow \quad \sin\phi = \frac{m_1 v_1 \sin\theta}{m_2 v_2} \quad (4)$$

From equation (1):

$$m_1 u - m_1 v_1 \cos\theta = m_2 v_2 \cos\phi \quad \Rightarrow \quad \cos\phi = \frac{m_1(u - v_1\cos\theta)}{m_2 v_2} \quad (5)$$

Now use $\sin^2\phi + \cos^2\phi = 1$:

Substitute from (4) and (5):

$$\left(\frac{m_1 v_1 \sin\theta}{m_2 v_2}\right)^2 + \left(\frac{m_1(u - v_1\cos\theta)}{m_2 v_2}\right)^2 = 1$$
$$\Rightarrow \frac{m_1^2}{m_2^2 v_2^2}\left[v_1^2\sin^2\theta + (u - v_1\cos\theta)^2\right] = 1$$
$$\Rightarrow v_2^2 = \frac{m_1^2}{m_2^2}\left[v_1^2\sin^2\theta + (u - v_1\cos\theta)^2\right] \quad (6)$$

Now substitute equation (6) into energy equation (3):

$$m_1u^2 = m_1v_1^2 + m_2v_2^2$$
$$\Rightarrow m_1u^2 - m_1v_1^2 = m_2v_2^2$$

Substitute v_2^2:

$$m_1u^2 - m_1v_1^2 = m_2 \cdot \frac{m_1^2}{m_2^2}\left[v_1^2\sin^2\theta + (u - v_1\cos\theta)^2\right]$$
$$\Rightarrow u^2 - v_1^2 = \frac{m_1}{m_2}\left[v_1^2\sin^2\theta + u^2 - 2uv_1\cos\theta + v_1^2\cos^2\theta\right]$$

Now expand the bracket and simplify:

$$u^2 - v_1^2 = \frac{m_1}{m_2}\left[u^2 - 2uv_1\cos\theta + v_1^2(\sin^2\theta + \cos^2\theta)\right]$$
$$\Rightarrow u^2 - v_1^2 = \frac{m_1}{m_2}\left[u^2 - 2uv_1\cos\theta + v_1^2\right]$$

Bring all terms to one side:

$$\left(1 - \frac{m_1}{m_2}\right)u^2 - \left(1 + \frac{m_1}{m_2}\right)v_1^2 + \frac{2m_1uv_1\cos\theta}{m_2} = 0 \quad (7)$$

Now solve for $\cos\theta$ from this equation:

$$\cos\theta = \frac{\left(1 + \frac{m_1}{m_2}\right)v_1^2 + \left(\frac{m_1}{m_2} - 1\right)u^2}{2\frac{m_1}{m_2}uv_1}$$

Let's denote $k = \frac{m_1}{m_2}$, then:

$$\cos\theta = \frac{(1+k)v_1^2 + (k-1)u^2}{2kuv_1}$$

To find the maximum θ, we minimize $\cos\theta$, since $\cos\theta$ decreases as θ increases.

Differentiate $\cos\theta$ with respect to v_1, set derivative to zero, and solve. This yields:

$$v_1 = u\sqrt{\frac{k-1}{k+1}} \quad (\text{if } k > 1)$$

Substitute back into the expression for $\cos\theta$ and simplify to get:

$$\cos\theta_{\text{max}} = \sqrt{1 - \left(\frac{m_2}{m_1}\right)^2} \quad \Rightarrow \quad \theta_{\text{max}} = \sin^{-1}\left(\frac{m_2}{m_1}\right) \quad \text{for } m_2 \leq m_1$$

If $m_2 > m_1$, the expression inside the arcsine becomes greater than 1, which is unphysical, so the maximum possible deflection is:

$$\theta_{\text{max}} = 90^\circ$$

Conclusion: The maximum possible deflection angle θ of particle P after an elastic collision with a stationary particle Q is given by:

$$\boxed{\theta_{\text{max}} = \sin^{-1}\left(\frac{m_2}{m_1}\right)} \quad \text{for } m_2 \leq m_1$$

and

$$\theta_{\text{max}} = 90^\circ \quad \text{for } m_2 > m_1$$

This result reflects the constraint imposed by conservation laws and the geometry of elastic collisions.

20 A planet revolves around the Sun in an elliptic orbit of eccentricity e. If T is the time period of the planet, find the time spent by the planet between the ends of the minor axis close to the Sun.

Introduction: A planet moves around the Sun in an elliptical orbit with eccentricity e, with the Sun at one focus. We need to find the time the planet takes to travel between the endpoints of the minor axis along the arc *closer* to the Sun (the segment passing through perihelion). We apply Kepler's Second Law: equal areas are swept out in equal times.

Kepler's Second Law: The areal velocity is constant:

$$\frac{dA}{dt} = \frac{\text{Total area}}{T} = \frac{\pi ab}{T}$$

where a is the semi-major axis and $b = a\sqrt{1-e^2}$ is the semi-minor axis.

Step 1: Geometry of the ellipse The endpoints of the minor axis occur at true anomalies:

$$\nu_1 = \frac{\pi}{2}, \quad \nu_2 = \frac{3\pi}{2}$$

assuming perihelion lies along the major axis at $\nu = 0$.

Step 2: Area swept from ν_1 to ν_2 The area swept between true anomalies ν_1 and ν_2 is:

$$A = \frac{a^2(1-e^2)}{2}\left[\nu_2 - \nu_1 + e(\sin\nu_2 - \sin\nu_1)\right]$$

For the arc from $\nu = \frac{\pi}{2}$ to $\nu = \frac{3\pi}{2}$ (the far arc, through aphelion):

$$\begin{aligned} A_{\text{far}} &= \frac{a^2(1-e^2)}{2}\left[\pi + e(\sin\frac{3\pi}{2} - \sin\frac{\pi}{2})\right] \\ &= \frac{a^2(1-e^2)}{2}\left[\pi + e(-1-1)\right] \\ &= \frac{a^2(1-e^2)}{2}(\pi - 2e) \end{aligned}$$

Since $b^2 = a^2(1-e^2)$:

$$A_{\text{far}} = \frac{b^2}{2}(\pi - 2e)$$

Step 3: Time spent on the far arc Using Kepler's Second Law:

$$t_{\text{far}} = \frac{A_{\text{far}}}{\pi ab/T} = \frac{\frac{b^2}{2}(\pi - 2e)}{\pi ab/T}$$

Since $b = a\sqrt{1-e^2}$, we have $b^2 = a^2(1-e^2)$, so:

$$\frac{b^2}{ab} = \frac{a^2(1-e^2)}{ab} = \frac{a(1-e^2)}{b} = \frac{a\sqrt{1-e^2}}{b} = \frac{b}{b} = 1$$

Wait, let me recalculate this more carefully:

$$\frac{b^2}{ab} = \frac{b}{a}$$

Therefore:

$$t_{\text{far}} = \frac{T \cdot b(\pi - 2e)}{2\pi a} = \frac{T\sqrt{1-e^2}(\pi - 2e)}{2\pi}$$

Actually, let me use a cleaner approach. Since $b^2 = a^2(1-e^2)$:

$$t_{\text{far}} = \frac{A_{\text{far}} \cdot T}{\pi ab} = \frac{\frac{a^2(1-e^2)}{2}(\pi - 2e) \cdot T}{\pi ab} = \frac{T(\pi - 2e)}{2\pi} \cdot \frac{a(1-e^2)}{b}$$

Since $b = a\sqrt{1-e^2}$, we have $\frac{a(1-e^2)}{b} = \frac{a(1-e^2)}{a\sqrt{1-e^2}} = \sqrt{1-e^2}$

Therefore:

$$t_{\text{far}} = \frac{T(\pi - 2e)\sqrt{1-e^2}}{2\pi}$$

Step 4: Time on the near arc (closer to the Sun) The time on the near arc is:

$$t_{\text{near}} = T - t_{\text{far}} = T - \frac{T(\pi - 2e)\sqrt{1-e^2}}{2\pi}$$
$$= \frac{T}{2\pi}\left[2\pi - (\pi - 2e)\sqrt{1-e^2}\right]$$

Conclusion: The time spent by the planet between the endpoints of the minor axis along the arc closer to the Sun is:

$$\boxed{\Delta t = \frac{T}{2\pi}\left[2\pi - (\pi - 2e)\sqrt{1-e^2}\right]}$$

For small eccentricity, this reduces to approximately $\frac{T(\pi+2e)}{2\pi}$.