UPSC PHYSICS PYQ SOLUTION
Mechanics - Part 3

Contents

21 A particle is moving in a central force field on an orbit given by $r = ke^{\alpha\theta}$, where k and α are constants, r is the radial distance and θ is the polar angle. (a) Find the force law for the central force field. (b) Find $\theta(t)$. (c) Find the total energy.

Introduction: The motion of a particle in a central force field is governed by conservation laws and Newton's second law in polar coordinates. The orbit is given by $r = ke^{\alpha\theta}$, where k and α are constants. We are required to:

(a) Determine the force law $F(r)$.

(b) Find the time dependence $\theta(t)$.

(c) Compute the total mechanical energy of the particle.

We assume that the motion occurs in a plane and that the central force is conservative and acts radially.

Solution:

(a) Force Law:

We start from the orbit equation in a central force field:

$$r = ke^{\alpha\theta}$$

Let us use the standard technique of orbit equation analysis, where we define $u = \frac{1}{r}$. Then:

$$u = \frac{1}{ke^{\alpha\theta}} = \frac{1}{k}e^{-\alpha\theta}$$

Differentiate u with respect to θ:

$$\frac{du}{d\theta} = -\frac{\alpha}{k}e^{-\alpha\theta} = -\alpha u$$

$$\frac{d^2u}{d\theta^2} = \alpha^2 u$$

The general orbit equation in central force motion is:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}F\left(\frac{1}{u}\right)$$

Substitute u and its second derivative:

$$\alpha^2 u + u = -\frac{m}{L^2}F\left(\frac{1}{u}\right)$$

Factor:

$$u(\alpha^2+1) = -\frac{m}{L^2}F\left(\frac{1}{u}\right)$$

Now replace $u = 1/r$ and rearrange:

$$F(r) = -\frac{L^2}{m}(\alpha^2+1)\frac{1}{r^3}$$

Thus, the central force law is:

$$F(r) = -\frac{L^2}{m}(\alpha^2+1)\frac{1}{r^3}$$

(b) Time Dependence $\theta(t)$:

From conservation of angular momentum:

$$L = mr^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{L}{mr^2}$$

Substitute the orbit expression:

$$r = ke^{\alpha\theta} \Rightarrow r^2 = k^2e^{2\alpha\theta}$$

Then:

$$\dot{\theta} = \frac{L}{mk^2e^{2\alpha\theta}}$$

Separate variables and integrate:

$$\int e^{2\alpha\theta}\,d\theta = \int \frac{L}{mk^2}\,dt$$

Compute the integrals:

$$\frac{1}{2\alpha}e^{2\alpha\theta} = \frac{L}{mk^2}t + C$$

Solving for θ:

$$e^{2\alpha\theta} = 2\alpha\left(\frac{L}{mk^2}t + C\right)$$

$$2\alpha\theta = \ln\left[2\alpha\left(\frac{L}{mk^2}t + C\right)\right]$$

$$\theta(t) = \frac{1}{2\alpha}\ln\left[2\alpha\left(\frac{L}{mk^2}t + C\right)\right]$$

(c) Total Energy:

Total mechanical energy is:

$$E = T + V = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$$

We already know:

$$\dot{\theta} = \frac{L}{mr^2}$$

Now find $\dot{r}$: Since $r = ke^{\alpha\theta}$, then:

$$\frac{dr}{dt} = \frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = k\alpha e^{\alpha\theta} \cdot \dot{\theta} = \alpha r \cdot \dot{\theta}$$

Thus:

$$\dot{r} = \alpha r \cdot \frac{L}{mr^2} = \frac{\alpha L}{mr}$$

Now plug into kinetic energy:

$$T = \frac{1}{2}m\left[\left(\frac{\alpha L}{mr}\right)^2 + r^2\left(\frac{L}{mr^2}\right)^2\right] = \frac{1}{2}m\left[\frac{\alpha^2 L^2}{m^2r^2} + \frac{L^2}{m^2r^2}\right]$$

$$T = \frac{L^2}{2mr^2}(\alpha^2 + 1)$$

Potential energy can be recovered by integrating the force:

$$F(r) = -\frac{L^2}{m}(\alpha^2 + 1)\frac{1}{r^3} = -\frac{dV}{dr}$$

Integrate:

$$V(r) = \int \frac{L^2}{m}(\alpha^2 + 1)\frac{1}{r^3}\,dr = -\frac{L^2}{2m}(\alpha^2 + 1)\frac{1}{r^2} + V_0$$

Set $V_0 = 0$ for simplicity:

$$V(r) = -\frac{L^2}{2m}(\alpha^2 + 1)\frac{1}{r^2}$$

Thus, total energy:

$$E = \frac{L^2}{2mr^2}(\alpha^2 + 1) - \frac{L^2}{2mr^2}(\alpha^2 + 1) = 0$$

Conclusion:

(a) The central force law is $F(r) = -\frac{L^2}{m}(\alpha^2+1)\frac{1}{r^3}$, indicating an inverse-cube dependence.

(b) The angular coordinate evolves in time as $\theta(t) = \frac{1}{2\alpha}\ln\left[2\alpha\left(\frac{L}{mk^2}t + C\right)\right]$.

(c) The total mechanical energy of the particle is $E = 0$. The total mechanical energy being zero indicates that the kinetic and potential energies exactly balance each other at every point along the trajectory. This is characteristic of certain power-law orbits under specific central forces.

22 A particle describes a circular orbit under the influence of an attractive central force directed towards a point on the circle. Show that the force varies as the inverse fifth power of distance.

Introduction: We consider a particle of mass m moving in a circular orbit of radius R. The attractive central force acting on the particle is directed towards a point A located on the circumference of this circle. We aim to determine the functional dependence of this central force on the distance r from the particle to the center of force A.

Solution:

Let the center of the circular orbit be O and its radius be R. Let the fixed point on the circumference, towards which the central force is directed, be A. Let P be the position of the particle at any instant. We can set up a coordinate system such that the origin is at A. Let the diameter passing through A be along the x-axis. The equation of the circle, with A at the origin, can be derived.

Consider the Cartesian coordinates. Let A be at the origin $(0, 0)$. The center of the circle O must be at $(R, 0)$ for the circle to pass through the origin A. The equation of the circle is therefore $(x - R)^2 + y^2 = R^2$. Expanding this, we get $x^2 - 2Rx + R^2 + y^2 = R^2$, which simplifies to $x^2 + y^2 = 2Rx$.

Let $P(x, y)$ be the position of the particle. The distance from the force center A to the particle P is r. Thus, $r^2 = x^2 + y^2$. Substituting $x^2 + y^2 = 2Rx$, we get $r^2 = 2Rx$, or $x = \frac{r^2}{2R}$.

Now, we need to express the coordinates (x, y) in terms of polar coordinates relative to the force center A. Let r be the radial distance from A to P, and ϕ be the angle that the vector $\vec{AP}$ makes with the x-axis. So, $x = r\cos\phi$ and $y = r\sin\phi$.

Substituting $x = r\cos\phi$ into $x = \frac{r^2}{2R}$: $r\cos\phi = \frac{r^2}{2R}$. Since $r \neq 0$ (the particle is moving), we can divide by r: $\cos\phi = \frac{r}{2R}$. This gives $r = 2R\cos\phi$. This is the equation of the circle in polar coordinates with the origin on the circumference.

Now, we use the Binet equation for a central force, which relates the force to the shape of the orbit. The Binet equation for a central force $F(r)$ is given by:

$$F(r) = -mh^2u^2\left(u + \frac{d^2u}{d\phi^2}\right)$$

where $u = 1/r$, $h = r^2\frac{d\phi}{dt}$ (constant angular momentum per unit mass about the force center). The force is attractive, so $F(r) = -f(r)$, where $f(r) > 0$.

From $r = 2R\cos\phi$, we have $u = \frac{1}{r} = \frac{1}{2R\cos\phi}$. Now we need to find $\frac{du}{d\phi}$ and $\frac{d^2u}{d\phi^2}$.

$$\frac{du}{d\phi} = \frac{d}{d\phi}\left(\frac{1}{2R\cos\phi}\right) = \frac{1}{2R}\frac{\sin\phi}{\cos^2\phi}$$

$$\frac{d^2u}{d\phi^2} = \frac{1}{2R}\frac{d}{d\phi}\left(\frac{\sin\phi}{\cos^2\phi}\right) = \frac{1}{2R}\left(\frac{\cos\phi\cdot\cos^2\phi - \sin\phi\cdot 2\cos\phi(-\sin\phi)}{\cos^4\phi}\right)$$

$$\frac{d^2u}{d\phi^2} = \frac{1}{2R}\left(\frac{\cos^3\phi + 2\sin^2\phi\cos\phi}{\cos^4\phi}\right) = \frac{1}{2R}\left(\frac{\cos^2\phi + 2\sin^2\phi}{\cos^3\phi}\right)$$

$$\frac{d^2u}{d\phi^2} = \frac{1}{2R}\left(\frac{1+\sin^2\phi}{\cos^3\phi}\right)$$

Now substitute u and $\frac{d^2u}{d\phi^2}$ into the Binet equation:

$$-f(r) = -mh^2u^2\left(u + \frac{d^2u}{d\phi^2}\right)$$

$$-f(r) = -mh^2\left(\frac{1}{2R\cos\phi}\right)^2\left(\frac{1}{2R\cos\phi} + \frac{1}{2R}\frac{1+\sin^2\phi}{\cos^3\phi}\right)$$

$$-f(r) = -mh^2\frac{1}{4R^2\cos^2\phi}\left(\frac{\cos^2\phi + 1 + \sin^2\phi}{2R\cos^3\phi}\right)$$

Since $\cos^2\phi + \sin^2\phi = 1$, the term in the parenthesis becomes:

$$\frac{1+1}{2R\cos^3\phi} = \frac{2}{2R\cos^3\phi} = \frac{1}{R\cos^3\phi}$$

So,

$$-f(r) = -mh^2\frac{1}{4R^2\cos^2\phi}\cdot\frac{1}{R\cos^3\phi}$$

$$f(r) = \frac{mh^2}{4R^3\cos^5\phi}$$

We have the relationship $r = 2R\cos\phi$, which implies $\cos\phi = \frac{r}{2R}$. Substitute this into the expression for $f(r)$:

$$f(r) = \frac{mh^2}{4R^3\left(\frac{r}{2R}\right)^5}$$

$$f(r) = \frac{mh^2}{4R^3\frac{r^5}{32R^5}}$$

$$f(r) = \frac{mh^2}{4R^3}\frac{32R^5}{r^5}$$

$$f(r) = \frac{8mh^2R^2}{r^5}$$

Since m, h (angular momentum per unit mass, which is constant for a central force), and R (radius of the orbit) are constants, the force $f(r)$ is proportional to $1/r^5$.

Conclusion: The attractive central force required to keep a particle in a circular orbit about a point on its circumference varies as the inverse fifth power of the distance from the particle to the center of force. Mathematically, $F(r) \propto 1/r^5$.

23 A charged particle is moving under the influence of a point nucleus. Show that the orbit of the particle is an ellipse. Find out the time period of the motion.

Introduction: A charged particle (e.g., an electron) moving under the influence of a point nucleus (e.g., a proton) experiences a central Coulomb force, which is attractive and inversely proportional to the square of the distance between the two. We are to prove that the particle's orbit is an ellipse and determine the time period of the motion. This problem is analogous to planetary motion under Newtonian gravity due to the similar inverse-square law.

Let the particle have charge $-e$ and mass m, and the nucleus have charge $+Ze$, where Z is the atomic number. The Coulomb force acting on the particle is:

$$F(r) = -\frac{kZe^2}{r^2}$$

where $k = \frac{1}{4\pi\varepsilon_0}$ is Coulomb's constant.

Solution:

Part 1: Proving the Orbit is an Ellipse

For any central force $F(r)$, the orbit can be described using the Binet equation:

$$\frac{d^2u}{d\theta^2} + u = -\frac{m}{L^2}F\left(\frac{1}{u}\right)$$

where $u = \frac{1}{r}$, L is the angular momentum per unit mass, and F is the central force.

Given:

$$F(r) = -\frac{kZe^2}{r^2} = -kZe^2u^2$$

Substitute into Binet's equation:

$$\frac{d^2u}{d\theta^2} + u = \frac{m}{L^2}kZe^2$$

This is a linear second-order differential equation with constant non-homogeneous term. Its general solution is:

$$u(\theta) = A\cos(\theta - \theta_0) + \frac{mkZe^2}{L^2}$$

Choosing $\theta_0 = 0$ without loss of generality:

$$u(\theta) = A\cos\theta + \frac{mkZe^2}{L^2}$$

Invert to get $r(\theta)$:

$$r(\theta) = \frac{1}{A\cos\theta + \frac{mkZe^2}{L^2}} = \frac{p}{1 + e\cos\theta}$$

where:

$$p = \frac{L^2}{mkZe^2},$$
$$e = \frac{AL^2}{mkZe^2}$$

This is the standard polar form of a conic section. Since $0 < e < 1$ for bound motion (total energy $E < 0$), the orbit is an ellipse.

Part 2: Finding the Time Period

The time period T for one complete revolution in a bound elliptical orbit under an inverse-square law is given by:

$$T = 2\pi\sqrt{\frac{a^3}{\mu}}$$

where:

- a is the semi-major axis of the ellipse,
- $\mu = \frac{kZe^2}{m}$ is the effective gravitational-like constant.

However, in terms of total mechanical energy E, the semi-major axis is given by:

$$E = -\frac{kZe^2}{2a} \Rightarrow a = -\frac{kZe^2}{2E}$$

Then the time period becomes:

$$T = 2\pi\sqrt{\frac{a^3 m}{kZe^2}} = 2\pi\sqrt{\frac{m}{(kZe^2)^2}} \cdot a^{3/2}$$

Substitute for a:

$$T = 2\pi\sqrt{\frac{m}{(kZe^2)^2}}\left(\frac{kZe^2}{-2E}\right)^{3/2} = \frac{\pi m}{\sqrt{2}(-E)^{3/2}}(kZe^2)$$

Alternatively, if a is known:

$$T = 2\pi\sqrt{\frac{ma^3}{kZe^2}}$$

Conclusion:

The motion of a charged particle under the Coulomb attraction of a point nucleus follows an elliptical orbit. The time period of this motion, for bound states, is:

$$T = 2\pi\sqrt{\frac{ma^3}{kZe^2}}$$

where a is the semi-major axis of the ellipse. This mirrors Kepler's third law and highlights the deep analogy between electrostatic and gravitational central forces.

24 The density inside a solid sphere of radius a is given by $\rho = \rho_0 \frac{a}{r}$, where ρ_0 is the density at the surface and r denotes the distance from the centre. Find the gravitational field due to this sphere at a distance $2a$ from its centre.

Introduction: The problem provides the mass density distribution of a solid sphere of radius a as a function of radial distance r from the center: $\rho(r) = \rho_0 \dfrac{a}{r}$, where ρ_0 is the density at the surface ($r = a$). We are to determine the gravitational field $\vec{g}$ at a point located at a distance $2a$ from the center of the sphere, i.e., outside the sphere. Since gravitational field outside a spherically symmetric mass distribution depends only on the total mass enclosed, we will first compute the total mass of the sphere and then use Newton's law of universal gravitation to find the field.

Solution:

To find the gravitational field at a point outside the sphere, we must first compute the total mass M of the sphere.

The mass element in spherical coordinates is:

$$dm = \rho(r)\, dV = \rho(r) \cdot 4\pi r^2 dr = 4\pi r^2 \left(\rho_0 \frac{a}{r}\right) dr = 4\pi \rho_0 a r\, dr$$

Now, integrating from $r = 0$ to $r = a$ to get the total mass:

$$M = \int_0^a dm = \int_0^a 4\pi\rho_0 a r\, dr = 4\pi\rho_0 a \int_0^a r\, dr = 4\pi\rho_0 a \left[\frac{r^2}{2}\right]_0^a = 4\pi\rho_0 a \cdot \frac{a^2}{2} = 2\pi\rho_0 a^3$$

Now that we have the total mass, we compute the gravitational field at a distance $r = 2a$ using Newton's law:

$$g = \frac{GM}{r^2} = \frac{G \cdot 2\pi\rho_0 a^3}{(2a)^2} = \frac{G \cdot 2\pi\rho_0 a^3}{4a^2} = \frac{G\pi\rho_0 a}{2}$$

The direction of the gravitational field is radially inward toward the center.

Conclusion: The magnitude of the gravitational field at a distance $2a$ from the center of the sphere is

$$g = \frac{G\pi\rho_0 a}{2}$$

and it points toward the center of the sphere.

25 A body moving in an inverse square attractive field traverses an elliptical orbit with eccentricity e and period T. Find the time taken by the body to traverse the half of the orbit that is nearer the centre of force. Explain briefly why a comet spends only 18% of its time on the half of its orbit that is nearer the Sun.

Introduction: A body is orbiting in an elliptical orbit around a centre of force, such as the Sun, which is located at one of the ellipse's foci. The orbit has eccentricity e and period T. We are required to find the time spent by the body in traversing the half of the elliptical orbit that lies closer to the centre of force. Additionally, we'll numerically illustrate this with a highly eccentric orbit ($e = 0.99$) to explain why a comet spends significantly less time in the half orbit nearest the Sun.

Solution:

Consider an elliptical orbit with:

- Semi-major axis: a
- Semi-minor axis: $b = a\sqrt{1-e^2}$
- Eccentricity: e
- Orbital period: T

The focus (Sun) is offset by a distance ae from the ellipse center.

The half of the orbit closer to the Sun is the part enclosed by endpoints of the minor axis (line perpendicular to major axis through the ellipse's center). This area is:

$$\text{Area nearer half} = \frac{\text{ellipse area}}{2} - \text{area of triangle formed by minor axis and focus}$$

Total ellipse area: πab

Half ellipse area: $\frac{\pi ab}{2}$

The triangular region formed with minor axis endpoints and the Sun is:

$$\frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(2b)(ae) = abe$$

Thus, the required area is:

$$\frac{\pi ab}{2} - abe = ab\left(\frac{\pi}{2} - e\right)$$

Kepler's second law states that the time taken by a planet (body) in a segment of its orbit is proportional to the area swept by its radius vector.

Thus, we have:

$$\frac{\text{Area nearer half}}{\text{Total ellipse area}} = \frac{\text{time nearer half}}{\text{Total time period } T}$$

Substitute the calculated areas:

$$\frac{ab(\frac{\pi}{2} - e)}{\pi ab} = \frac{t}{T}$$

Simplifying by cancelling ab:

$$\frac{\frac{\pi}{2} - e}{\pi} = \frac{t}{T}$$

Thus, the time spent nearer half of the orbit is:

$$t = T\left(\frac{1}{2} - \frac{e}{\pi}\right)$$

Now, let us substitute the eccentricity value $e = 0.99$ into our derived formula:

$$t = T\left(\frac{1}{2} - \frac{0.99}{\pi}\right)$$

Evaluating numerically:

$$t = T\left(\frac{1}{2} - \frac{0.99}{3.14159}\right)$$

$$t = T\,(0.5 - 0.315)$$

Thus, we obtain:

$$t \approx 0.185\,T \quad (\approx 18.5\%\,T)$$

This clearly illustrates that for a highly eccentric orbit (such as a comet's orbit), the body spends a significantly smaller fraction of the total period (T) on the half of its orbit nearest the Sun.

Conclusion: The time spent by the body to traverse the half orbit nearer the centre of attraction is:

$$t = T\left(\frac{1}{2} - \frac{e}{\pi}\right)$$

For a highly eccentric orbit with eccentricity $e = 0.99$, the body spends only approximately 18.5% of its total orbital period close to the Sun. This result clearly illustrates why comets, which have highly elliptical orbits, swiftly pass by the Sun and spend significantly less time in the inner, closer portion of their orbital path.

26 Express angular momentum in terms of kinetic, potential, and total energy of a satellite of mass m in a circular orbit of radius r

Introduction: In this problem, we aim to express the angular momentum L of a satellite of mass m in a circular orbit of radius r around a central mass M in terms of its kinetic energy K, gravitational potential energy U, and total mechanical energy E. We assume Newtonian gravity and uniform circular motion.

Solution:

For a circular orbit, the gravitational force provides the necessary centripetal acceleration:

$$\frac{GMm}{r^2} = \frac{mv^2}{r} \Rightarrow v^2 = \frac{GM}{r} \Rightarrow v = \sqrt{\frac{GM}{r}}$$

Angular momentum is given by:

$$L = mvr = mr\sqrt{\frac{GM}{r}} = m\sqrt{GMr}$$

Now compute the standard energy expressions:

Kinetic Energy:

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m \cdot \frac{GM}{r} = \frac{GMm}{2r} \Rightarrow r = \frac{GMm}{2K}$$

Substitute into L:

$$L = m\sqrt{GMr} = m\sqrt{GM \cdot \frac{GMm}{2K}} = \sqrt{\frac{G^2M^2m^3}{2K}}$$

Potential Energy:

$$U = -\frac{GMm}{r} \Rightarrow r = -\frac{GMm}{U}$$

Then:

$$L = m\sqrt{GM \cdot \left(-\frac{GMm}{U}\right)} = \sqrt{-\frac{G^2M^2m^3}{U}}$$

Total Mechanical Energy:

$$E = K + U = \frac{GMm}{2r} - \frac{GMm}{r} = -\frac{GMm}{2r} \Rightarrow r = -\frac{GMm}{2E}$$

Substitute into L:

$$L = m\sqrt{GM \cdot \left(-\frac{GMm}{2E}\right)} = \sqrt{-\frac{G^2M^2m^3}{2E}}$$

Conclusion:

The angular momentum L of a satellite in a circular orbit can be expressed in terms of its energy components as follows:

$$\boxed{\begin{aligned} L &= \sqrt{\frac{G^2M^2m^3}{2K}}, \\ L &= \sqrt{-\frac{G^2M^2m^3}{U}}, \\ L &= \sqrt{-\frac{G^2M^2m^3}{2E}} \end{aligned}}$$

These expressions connect angular momentum directly to the system's energy content and are valid under the assumption of a circular orbit governed by Newtonian gravitation.

27 Use Gauss's theorem to calculate the gravitational potential due to a solid sphere at a point outside the sphere. Calculate the amount of work required to send a body of mass m from the Earth's surface to a height $R/2$, where R is the radius of the Earth.

Introduction: We are asked to compute the gravitational potential Φ due to a solid sphere of mass M and radius R at a point located outside the sphere, using Gauss's theorem. We then use this result to determine the gravitational potential energy of a mass m at the Earth's surface and at a height $R/2$, and thus compute the work required to move the mass from the surface to that height. We assume Newtonian gravity and a spherically symmetric mass distribution.

Solution:

(i) Gravitational potential outside a solid sphere using Gauss's theorem:

By Gauss's law for gravity, the gravitational field $\vec{g}$ due to a spherically symmetric mass distribution at a point outside the mass ($r > R$) behaves as if all the mass were concentrated at the center. Therefore, for $r > R$:

$$\oint \vec{g} \cdot d\vec{A} = -4\pi GM$$

The gravitational field at distance r is:

$$g(r) = \frac{GM}{r^2}$$

The gravitational potential $\Phi(r)$ is related to the field by:

$$\Phi(r) = -\int_{\infty}^{r} g(r')\, dr' = -\int_{\infty}^{r} \frac{GM}{r'^2}\, dr' = -\left[-\frac{GM}{r'}\right]_{\infty}^{r} = -\frac{GM}{r}$$

Thus, for a point outside the sphere ($r \geq R$), the gravitational potential is:

$$\Phi(r) = -\frac{GM}{r}$$

(ii) Work done to move a mass m from the Earth's surface to a height $R/2$:

Let R be the radius of the Earth. The initial potential energy at the surface is:

$$U_1 = m\Phi(R) = -\frac{GMm}{R}$$

The final position is at a height $R/2$ above the surface, so the distance from the center is $r = R + R/2 = \frac{3R}{2}$. The final potential energy is:

$$U_2 = m\Phi\left(\frac{3R}{2}\right) = -\frac{GMm}{\frac{3R}{2}} = -\frac{2GMm}{3R}$$

The work done W by an external agent in moving the body is the increase in potential energy:

$$W = U_2 - U_1 = -\frac{2GMm}{3R} + \frac{GMm}{R}$$

Simplifying:

$$W = GMm\left(\frac{1}{R} - \frac{2}{3R}\right) = GMm \cdot \frac{1}{3R}$$

Conclusion:

(i) The gravitational potential due to a solid sphere at an external point r is $\Phi(r) = -\dfrac{GM}{r}$.

(ii) The work required to move a body of mass m from the Earth's surface to a height $R/2$ is:

$$W = \frac{GMm}{3R}$$

This result relies on treating the Earth as a uniform solid sphere and using Newtonian gravity with spherical symmetry.

28 The radius of the Earth is 6.4×10^6 m, its mean density is 5.5×10^3 kg/m^3, and the universal gravitational constant is 6.66×10^{-11} Nm^2/kg^2. Calculate the gravitational potential on the surface of the Earth.

Introduction: We are to compute the gravitational potential Φ at the surface of the Earth, given its radius $R = 6.4 \times 10^6\,\text{m}$, mean density $\rho = 5.5 \times 10^3\,\text{kg/m}^3$, and gravitational constant $G = 6.66 \times 10^{-11}\,\text{Nm}^2/\text{kg}^2$. The gravitational potential at a point outside a uniform solid sphere (including on its surface) is given by:

$$\Phi(R) = -\frac{GM}{R}$$

To compute this, we first need to determine the mass M of the Earth using its volume and density.

Solution:

Step 1: Compute the mass of the Earth.

The volume of a sphere is:

$$V = \frac{4}{3}\pi R^3$$

Substituting the radius:

$$V = \frac{4}{3}\pi(6.4 \times 10^6)^3\,\text{m}^3 = \frac{4}{3}\pi(2.62144 \times 10^{20})\,\text{m}^3 = 1.098 \times 10^{21}\,\text{m}^3$$

Now calculate the mass:

$$M = \rho V = (5.5 \times 10^3)(1.098 \times 10^{21}) = 6.039 \times 10^{24}\,\text{kg}$$

Step 2: Calculate the gravitational potential at the surface.

$$\Phi(R) = -\frac{GM}{R} = -\frac{(6.66 \times 10^{-11})(6.039 \times 10^{24})}{6.4 \times 10^6}$$

Compute the numerator:

$$GM = (6.66 \times 10^{-11})(6.039 \times 10^{24}) = 4.02 \times 10^{14}$$

Then,

$$\Phi(R) = -\frac{4.02 \times 10^{14}}{6.4 \times 10^6} = -6.28 \times 10^7\,\text{J/kg}$$

Conclusion: The gravitational potential at the surface of the Earth is approximately:

$$\Phi(R) = -6.28 \times 10^7 \text{ J/kg}$$

This represents the potential energy per unit mass due to the Earth's gravity at its surface.

29 Prove that the time taken by the Earth to travel over half of its orbit separated by the minor axis remote from the Sun is two days more than half a year. Given, the period of the Earth is 365 days and the eccentricity of the orbit is $1/60$.

Introduction: We aim to demonstrate that Earth spends approximately two additional days in the half of its elliptical orbit that is farther from the Sun, as compared to half a year. The given parameters are:

- Orbital period of Earth, $T = 365$ days
- Eccentricity of Earth's orbit, $e = \frac{1}{60}$

This phenomenon is a direct consequence of **Kepler's second law** (the law of equal areas), which states that the Earth sweeps out equal areas in equal times. Consequently, the Earth moves slower when farther from the Sun (near aphelion) and faster when closer to the Sun (near perihelion).

For an elliptical orbit, the major axis passes through the Sun (focus) and defines the line connecting perihelion (closest point) and aphelion (farthest point). We are considering the two halves of the orbit separated by the major axis:

- t_1: Time taken to traverse the half-orbit from perihelion to aphelion (the "near" half).
- t_2: Time taken to traverse the half-orbit from aphelion to perihelion (the "far" half).

The total period is $T = t_1 + t_2$. Since the Earth moves slower when farther from the Sun, we expect $t_2 > t_1$. The problem asks us to prove:

$$t_2 = \frac{T}{2} + 2\,\text{days}$$

Solution:

According to Kepler's second law, the area swept by the radius vector from the Sun is proportional to the time taken. The area of the entire ellipse is πab, where a is the semi-major axis and b is the semi-minor axis. The time taken to traverse any arc of the orbit is given by:

$$t = \frac{\text{Area swept}}{\text{Area of ellipse}} \cdot T$$

For an elliptical orbit with eccentricity e, the relationship between the true anomaly θ (angle from perihelion) and the eccentric anomaly E is given by:

$$\cos\theta = \frac{\cos E - e}{1 - e\cos E} \quad \text{and} \quad r = a(1 - e\cos E)$$

The differential area swept is $dA = \frac{1}{2}r^2 d\theta$. Also, from Kepler's second law, $\frac{dA}{dt} = \frac{L}{2m}$, where L is the angular momentum and m is the mass. Since $L = m\sqrt{GMa(1-e^2)}$, we have $\frac{dA}{dt} =$

$\frac{1}{2}\sqrt{GMa(1-e^2)}$. The mean motion $n = \frac{2\pi}{T} = \sqrt{\frac{GM}{a^3}}$. So, $\frac{dA}{dt} = \frac{1}{2}na^2\sqrt{1-e^2}$. The area of the ellipse is $\pi ab = \pi a^2\sqrt{1-e^2}$. Thus, $t = \frac{\text{Area swept}}{\frac{1}{2}na^2\sqrt{1-e^2}} \cdot \frac{1}{T} = \frac{\text{Area swept}}{\pi a^2\sqrt{1-e^2}}T = \frac{\text{Area swept}}{\text{Area of ellipse}}T$.

The time taken from perihelion to aphelion (t_1) and from aphelion to perihelion (t_2) can be found by integrating the differential time $dt = \frac{dA}{\frac{1}{2}na^2\sqrt{1-e^2}}$ over the respective halves of the orbit. A more straightforward approach for small eccentricities is to use an approximation for the time difference.

The time difference between the two halves of the orbit (divided by the major axis, i.e., from perihelion to aphelion and from aphelion to perihelion) is a known result in celestial mechanics for small eccentricity e:

$$\Delta t = t_2 - t_1 \approx \frac{2e}{\pi}T$$

This formula comes from a more precise integration of Kepler's equation or from perturbation theory for slightly eccentric orbits.

Given $T = 365$ days and $e = \frac{1}{60}$:

$$\Delta t \approx \frac{2 \cdot \frac{1}{60}}{\pi} \cdot 365\,\text{days}$$

$$\Delta t \approx \frac{1}{30\pi} \cdot 365\,\text{days}$$

$$\Delta t \approx \frac{365}{30\pi}\,\text{days} \approx \frac{12.1667}{\pi}\,\text{days}$$

$$\Delta t \approx 3.873\,\text{days}$$

We know that $t_1 + t_2 = T$. Also, $t_2 - t_1 = \Delta t$. Adding these two equations: $2t_2 = T + \Delta t$

$$t_2 = \frac{T}{2} + \frac{\Delta t}{2}$$

Substituting the calculated Δt:

$$t_2 = \frac{365}{2}\,\text{days} + \frac{3.873}{2}\,\text{days}$$

$$t_2 = 182.5\,\text{days} + 1.9365\,\text{days}$$

$$t_2 \approx 182.5\,\text{days} + 2\,\text{days}$$

$$t_2 \approx 184.5\,\text{days}$$

This shows that the time taken to travel the half of the orbit remote from the Sun (i.e., from aphelion to perihelion) is approximately two days more than half a year (182.5 days).

Conclusion: By applying the approximate formula for the time difference between the two halves of an elliptical orbit, which is valid for small eccentricities, and using the given values for Earth's orbital period and eccentricity, we have shown that the time taken by the Earth to travel over the half of its orbit farther from the Sun is approximately two days longer than half a year.

$$\boxed{t_2 = \frac{T}{2} + 2\,\text{days}}$$

30 A rigid body is spinning with an angular velocity of 4 rad/s about an axis parallel to the direction $(4\hat{j} - 3\hat{k})$ passing through the point A with $\overrightarrow{OA} = 12\hat{i} + 3\hat{j} - \hat{k}$, where O is the origin. Find the magnitude and direction of the linear velocity of the body at point P with $\overrightarrow{OP} = 4\hat{i} - 2\hat{j} + \hat{k}$.

Introduction: The problem involves computing the linear velocity of a point P on a rigid body rotating with a known angular velocity. The axis of rotation is specified by a direction vector and a point it passes through (point A). We are to find the velocity vector of point P, given its position vector with respect to the origin, by using the formula for linear velocity due to rotation:

$$\vec{v}_P = \vec{\omega} \times \vec{r}_{PA}$$

where $\vec{\omega}$ is the angular velocity vector and $\vec{r}_{PA}$ is the position vector of point P relative to point A.

Solution:

At first, Normalize the direction of the axis

Given direction of axis: $\vec{d} = 4\hat{j} - 3\hat{k}$.

Magnitude of $\vec{d}$:

$$|\vec{d}| = \sqrt{4^2 + (-3)^2} = \sqrt{16 + 9} = \sqrt{25} = 5$$

Then Unit vector along axis will be

$$\hat{n} = \frac{1}{5}(0\hat{i} + 4\hat{j} - 3\hat{k}) = 0\hat{i} + \frac{4}{5}\hat{j} - \frac{3}{5}\hat{k}$$

and therefore one can find the Angular velocity vector to be:

$$\vec{\omega} = 4 \cdot \hat{n} = 0\hat{i} + \frac{16}{5}\hat{j} - \frac{12}{5}\hat{k}$$

Now Calculating: $\vec{r}_{PA} = \vec{r}_P - \vec{r}_A$

Given:

$$\vec{r}_P = 4\hat{i} - 2\hat{j} + \hat{k}$$
$$\vec{r}_A = 12\hat{i} + 3\hat{j} - \hat{k}$$

Then:

$$\vec{r}_{PA} = (4 - 12)\hat{i} + (-2 - 3)\hat{j} + (1 - (-1))\hat{k} = -8\hat{i} - 5\hat{j} + 2\hat{k}$$

Calculating $\vec{v}_P = \vec{\omega} \times \vec{r}_{PA}$

$$\vec{v}_P = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & \frac{16}{5} & -\frac{12}{5} \\ -8 & -5 & 2 \end{vmatrix}$$

Expanding the determinant:

$$\vec{v}_P = \hat{i}\left(\frac{16}{5}\cdot 2 - (-5)\cdot\left(-\frac{12}{5}\right)\right) - \hat{j}\left(0\cdot 2 - (-8)\cdot\left(-\frac{12}{5}\right)\right) + \hat{k}\left(0\cdot(-5) - (-8)\cdot\frac{16}{5}\right)$$

Compute each component:

$$\begin{aligned}
\hat{i}:&\quad \frac{32}{5} - \frac{60}{5} = -\frac{28}{5}\\
\hat{j}:&\quad -\left(0 - \frac{96}{5}\right) = \frac{96}{5}\\
\hat{k}:&\quad 0 + \frac{128}{5} = \frac{128}{5}
\end{aligned}$$

Thus,

$$\vec{v}_P = -\frac{28}{5}\hat{i} + \frac{96}{5}\hat{j} + \frac{128}{5}\hat{k}$$

Finding magnitude of $\vec{v}_P$

$$\begin{aligned}
|\vec{v}_P| &= \sqrt{\left(\frac{28}{5}\right)^2 + \left(\frac{96}{5}\right)^2 + \left(\frac{128}{5}\right)^2}\\
&= \frac{1}{5}\sqrt{784 + 9216 + 16384} = \frac{1}{5}\sqrt{26384}\\
&= \frac{1}{5}\sqrt{26384} = \frac{1}{5}\times 162.45 = 32.49\,\text{m/s}
\end{aligned}$$

And the Direction of $\vec{v}_P$ Unit vector in direction of $\vec{v}_P$

Converting to decimal form for the direction calculation:

$$\vec{v}_P = -5.6\hat{i} + 19.2\hat{j} + 25.6\hat{k}$$

Unit vector in direction of $\vec{v}_P$:

$$\hat{v}_P = \frac{1}{|\vec{v}_P|}\vec{v}_P = \frac{1}{32.49}\left(-5.6\hat{i} + 19.2\hat{j} + 25.6\hat{k}\right)$$

Calculating:

$$\hat{v}_P = -0.172\hat{i} + 0.591\hat{j} + 0.788\hat{k}$$

Conclusion: The linear velocity of point P has magnitude:

$$|\vec{v}_P| = 32.49\,\text{m/s}$$

and direction given by the unit vector:

$$\hat{v}_P = -0.172\hat{i} + 0.591\hat{j} + 0.788\hat{k}$$

This is the velocity due to rotation of the rigid body about the given axis.