UPSC PHYSICS PYQ SOLUTION
Mechanics - Part 4

Contents

31 Suppose an $S^{'}$ frame is rotating with respect to a fixed frame having the same origin. Assume that the angular velocity $\vec{\omega}$ of the $S^{'}$-frame is given by $\vec{\omega} = 2t\hat{i} - t^2\hat{j} + (2t + 4)\hat{k}$, where t is time. The position vector $\vec{r}$ of a particle in $S^{'}$ at time t is $\vec{r} = (t^2 + 1)\hat{i} - 6t\hat{j} + 4t^3\hat{k}$. Calculate the Coriolis acceleration at $t = 1$ second.

Introduction: In this problem, we are given a rotating frame S' with angular velocity $\vec{\omega}$ relative to an inertial frame. The angular velocity is time-dependent, and the position vector $\vec{r}$ of a particle is given as a function of time in the rotating frame. We are to compute the Coriolis acceleration at time $t = 1\,\text{s}$. The Coriolis acceleration is given by the standard expression in non-inertial rotating frames:

$$\vec{a}_{\text{Coriolis}} = -2\vec{\omega} \times \vec{v}_{\text{rel}},$$

where $\vec{v}_{\text{rel}} = \frac{d\vec{r}}{dt}$ is the velocity of the particle in the rotating frame.

Solution: We begin by computing the relative velocity $\vec{v}_{\text{rel}}$ by differentiating the given position vector with respect to time:

$$\vec{r}(t) = (t^2 + 1)\hat{i} - 6t\hat{j} + 4t^3\hat{k},$$

so

$$\vec{v}_{\text{rel}} = \frac{d\vec{r}}{dt} = 2t\hat{i} - 6\hat{j} + 12t^2\hat{k}.$$

At $t = 1$ second, we evaluate:

$$\begin{aligned}\vec{v}_{\text{rel}}(1) &= 2(1)\hat{i} - 6\hat{j} + 12(1)^2\hat{k}\\ &= 2\hat{i} - 6\hat{j} + 12\hat{k}.\end{aligned}$$

Next, evaluate the angular velocity at $t = 1$:

$$\begin{aligned}\vec{\omega}(t) &= 2t\hat{i} - t^2\hat{j} + (2t + 4)\hat{k},\\ \vec{\omega}(1) &= 2\hat{i} - 1\hat{j} + 6\hat{k}.\end{aligned}$$

Now compute the Coriolis acceleration:

$$\vec{a}_{\text{Coriolis}} = -2\vec{\omega} \times \vec{v}_{\text{rel}}.$$

Compute the cross product:

$$\begin{aligned}\vec{\omega} \times \vec{v}_{\text{rel}} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 6 \\ 2 & -6 & 12 \end{vmatrix}\\ &= \hat{i}((-1)(12) - (6)(-6)) - \hat{j}((2)(12) - (6)(2)) + \hat{k}((2)(-6) - (-1)(2))\\ &= \hat{i}(-12 + 36) - \hat{j}(24 - 12) + \hat{k}(-12 + 2)\\ &= 24\hat{i} - 12\hat{j} - 10\hat{k}.\end{aligned}$$

Therefore,

$$\begin{aligned}\vec{a}_{\text{Coriolis}} &= -2(24\hat{i} - 12\hat{j} - 10\hat{k}) \\ &= -48\hat{i} + 24\hat{j} + 20\hat{k}.\end{aligned}$$

Conclusion: The Coriolis acceleration of the particle at $t = 1$ s is given by:

$$\vec{a}_{\text{Coriolis}} = -48\hat{i} + 24\hat{j} + 20\hat{k} \quad \text{(in SI units, m/s}^2\text{).}$$

This acceleration arises due to the rotation of the reference frame and its magnitude and direction are influenced both by the angular velocity vector and the relative velocity of the particle within the rotating frame.

32 Calculate the horizontal component of the Coriolis force acting on a body of mass 0.1 kg moving northward with a horizontal velocity of 100 m/s at $30^\circ N$ latitude on Earth.

Introduction: We are asked to compute the horizontal component of the Coriolis force acting on a body of mass $m = 0.1\,\text{kg}$ that is moving northward with a speed of $v = 100\,\text{m/s}$ at a geographic latitude of 30° north. The Coriolis force is given by:

$$\vec{F}_{\text{Coriolis}} = -2m\vec{\Omega} \times \vec{v},$$

where $\vec{\Omega}$ is the angular velocity vector of Earth and $\vec{v}$ is the velocity of the object in the rotating Earth frame. We are interested in the horizontal component of this force, which corresponds to the eastward deflection due to northward motion.

Solution: The angular velocity of Earth is:

$$\Omega = 7.292 \times 10^{-5}\,\text{rad/s}.$$

Let us denote:

- $\phi = 30^\circ$ as the latitude,
- $\vec{v} = v\hat{j}$ as northward velocity (assuming a local coordinate system where $\hat{i}$ is east, $\hat{j}$ is north, and $\hat{k}$ is up).

In this coordinate system, the angular velocity vector of Earth is:

$$\vec{\Omega} = \Omega\cos\phi\hat{j} + \Omega\sin\phi\hat{k}.$$

We compute the Coriolis acceleration using:

$$\vec{a}_{\text{Coriolis}} = -2\vec{\Omega} \times \vec{v}.$$

Substituting $\vec{v} = v\hat{j}$ and noting that $\hat{j} \times \hat{j} = 0$:

$$\begin{aligned}
\vec{\Omega} \times \vec{v} &= (\Omega\cos\phi\hat{j} + \Omega\sin\phi\hat{k}) \times (v\hat{j}) \\
&= \Omega\cos\phi\hat{j} \times v\hat{j} + \Omega\sin\phi\hat{k} \times v\hat{j} \\
&= 0 + \Omega v\sin\phi(\hat{k} \times \hat{j}) \\
&= -\Omega v\sin\phi\,\hat{i}.
\end{aligned}$$

Thus,

$$\vec{a}_{\text{Coriolis}} = -2\vec{\Omega} \times \vec{v} = 2\Omega v\sin\phi\,\hat{i}.$$

Now compute the Coriolis force:

$$F_{\text{Coriolis, horiz}} = m \cdot a_{\text{Coriolis}} = 2m\Omega v\sin\phi.$$

Substitute numerical values:

$$\begin{aligned} F_{\text{Coriolis, horiz}} &= 2 \cdot 0.1\,\text{kg} \cdot (7.292 \times 10^{-5}\,\text{rad/s}) \cdot 100\,\text{m/s} \cdot \sin(30^\circ) \\ &= 0.2 \cdot 7.292 \times 10^{-5} \cdot 100 \cdot 0.5 \\ &= 7.292 \times 10^{-4}\,\text{N}. \end{aligned}$$

Conclusion: The horizontal component of the Coriolis force acting on the body is approximately:

$$F_{\text{Coriolis, horiz}} = 7.292 \times 10^{-4}\,\text{N} \quad \text{(eastward direction).}$$

This force causes a small but measurable eastward deflection of northward-moving objects on the rotating Earth.

33 Derive the expression for Coriolis force and show that this force is perpendicular to the velocity and to the axis of rotation. What is the nature of this force?

Introduction: The Coriolis force arises in a rotating frame of reference and affects the motion of objects moving within such a frame. It is an apparent or fictitious force due to the non-inertial nature of the rotating system. The goal here is to derive the mathematical expression for the Coriolis force and demonstrate that it is always perpendicular to both the velocity of the object and the axis of rotation. We also aim to describe the nature (type and properties) of this force.

Solution: Consider a non-inertial frame S' rotating with angular velocity vector $\vec{\Omega}$ relative to an inertial frame S. Let a particle have a position vector $\vec{r}$ and a velocity $\vec{v}_{\text{rel}}$ as measured in the rotating frame.

The total acceleration of the particle as seen from the inertial frame is:

$$\vec{a}_{\text{abs}} = \left(\frac{d\vec{v}}{dt}\right)_{\text{inertial}}.$$

In a rotating frame, this is related to the observed acceleration in the rotating frame by:

$$\vec{a}_{\text{abs}} = \vec{a}_{\text{rel}} + 2\vec{\Omega} \times \vec{v}_{\text{rel}} + \vec{\Omega} \times (\vec{\Omega} \times \vec{r}) + \frac{d\vec{\Omega}}{dt} \times \vec{r},$$

where:

(i) $\vec{a}_{\text{rel}} = \left(\frac{d\vec{v}}{dt}\right)_{\text{rot}}$ is the acceleration in the rotating frame,

(ii) $2\vec{\Omega} \times \vec{v}_{\text{rel}}$ is the Coriolis acceleration,

(iii) $\vec{\Omega} \times (\vec{\Omega} \times \vec{r})$ is the centrifugal acceleration,

(iv) $\frac{d\vec{\Omega}}{dt} \times \vec{r}$ is the Euler acceleration (present if $\vec{\Omega}$ is time-dependent).

We isolate the Coriolis acceleration term, which leads to the Coriolis force:

$$\vec{F}_{\text{Coriolis}} = -2m\vec{\Omega} \times \vec{v}_{\text{rel}}.$$

Now let us analyze its direction. The cross product $\vec{\Omega} \times \vec{v}_{\text{rel}}$ is by definition perpendicular to both $\vec{\Omega}$ and $\vec{v}_{\text{rel}}$. Hence,

$$\vec{F}_{\text{Coriolis}} \perp \vec{\Omega}, \quad \vec{F}_{\text{Coriolis}} \perp \vec{v}_{\text{rel}}.$$

This is a fundamental property of the cross product: the resulting vector lies in a direction orthogonal to both operands.

To illustrate this geometrically:

- Let $\vec{v}_{\text{rel}}$ be a velocity vector tangent to the particle's path.
- Let $\vec{\Omega}$ represent the angular velocity vector of the rotating frame (aligned with the axis of rotation).

- Then $\vec{F}_{\text{Coriolis}} = -2m\vec{\Omega} \times \vec{v}_{\text{rel}}$ points in a direction perpendicular to the plane formed by $\vec{\Omega}$ and $\vec{v}_{\text{rel}}$.

Nature of the Coriolis Force:

(a) It is a **fictitious or inertial force**, arising due to the non-inertial nature of the rotating frame.

(b) It does not do work since it is always perpendicular to the velocity of the object.

(c) It alters the trajectory of the object, causing a deflection (to the right in the northern hemisphere and to the left in the southern hemisphere on Earth).

(d) It is proportional to the mass of the object, the speed in the rotating frame, and the angular speed of the rotation.

Conclusion: The Coriolis force is given by:

$$\vec{F}_{\text{Coriolis}} = -2m\vec{\Omega} \times \vec{v}_{\text{rel}}.$$

It acts perpendicular to both the velocity vector $\vec{v}_{\text{rel}}$ and the axis of rotation $\vec{\Omega}$, and is a fictitious force present in rotating frames. While it does not do work, it significantly influences motion in geophysical and astrophysical systems.

34 Consider two frames of reference S and $S^{'}$ having a common origin O. The frame $S^{'}$ is rotating with respect to the fixed frame S with uniform $\vec{\omega} = 3\hat{a}_x$ rad/s. A projectile of unit mass at position $\vec{r} = 7\hat{a}_x + 4\hat{a}_y$ m is moving with $\vec{v} = 14\hat{a}_x$ m/s. Calculate in the rotating frame $S^{'}$ the following forces on the projectile: (i) Euler's force (ii) Coriolis force (iii) Centrifugal force.

Introduction: We are given two frames of reference: an inertial frame S and a rotating frame S' with a common origin. The rotating frame has a constant angular velocity $\vec{\omega} = 3\hat{a}_x$ rad/s. A projectile of unit mass ($m = 1$) is located at position $\vec{r} = 7\hat{a}_x + 4\hat{a}_y$ m with velocity $\vec{v} = 14\hat{a}_x$ m/s in the inertial frame.

We are to calculate the following fictitious forces acting on the projectile in the rotating frame S':

(i) Euler's force: $-m\,\frac{d\vec{\omega}}{dt} \times \vec{r}$,

(ii) Coriolis force: $-2m\,\vec{\omega} \times \vec{v}_{\text{rel}}$,

(iii) Centrifugal force: $-m\,\vec{\omega} \times (\vec{\omega} \times \vec{r})$.

Solution:

Given: $m = 1$ kg (unit mass) $\vec{\omega} = 3\hat{a}_x$ rad/s $\vec{r} = 7\hat{a}_x + 4\hat{a}_y$ m $\vec{v} = 14\hat{a}_x$ m/s

(i) Euler's Force: The Euler's force is given by $\vec{F}_{\text{Euler}} = -m\,\frac{d\vec{\omega}}{dt} \times \vec{r}$. Since the angular velocity $\vec{\omega}$ is uniform (constant), its time derivative is zero:

$$\frac{d\vec{\omega}}{dt} = \frac{d}{dt}(3\hat{a}_x) = 0.$$

Therefore,

$$\vec{F}_{\text{Euler}} = -m\,(0) \times \vec{r} = \vec{0}.$$

(ii) Coriolis Force: The Coriolis force is given by $\vec{F}_{\text{Coriolis}} = -2m\vec{\omega} \times \vec{v}_{\text{rel}}$. First, we need to find the velocity of the particle in the rotating frame, $\vec{v}_{\text{rel}}$. The relationship between velocities in the inertial and rotating frames is:

$$\vec{v} = \vec{v}_{\text{rel}} + \vec{\omega} \times \vec{r}.$$

So,

$$\vec{v}_{\text{rel}} = \vec{v} - (\vec{\omega} \times \vec{r}).$$

Let's calculate $\vec{\omega} \times \vec{r}$:

$$\begin{aligned}
\vec{\omega} \times \vec{r} &= (3\hat{a}_x) \times (7\hat{a}_x + 4\hat{a}_y) \\
&= (3 \times 7)(\hat{a}_x \times \hat{a}_x) + (3 \times 4)(\hat{a}_x \times \hat{a}_y) \\
&= 0 + 12\hat{a}_z \\
&= 12\hat{a}_z.
\end{aligned}$$

Now, substitute this into the expression for $\vec{v}_{\text{rel}}$:

$$\vec{v}_{\text{rel}} = 14\hat{a}_x - 12\hat{a}_z.$$

Next, calculate $\vec{\omega} \times \vec{v}_{\text{rel}}$:

$$\begin{aligned}
\vec{\omega} \times \vec{v}_{\text{rel}} &= (3\hat{a}_x) \times (14\hat{a}_x - 12\hat{a}_z) \\
&= (3 \times 14)(\hat{a}_x \times \hat{a}_x) - (3 \times 12)(\hat{a}_x \times \hat{a}_z) \\
&= 0 - 36(-\hat{a}_y) \\
&= 36\hat{a}_y.
\end{aligned}$$

Finally, calculate the Coriolis force:

$$\vec{F}_{\text{Coriolis}} = -2m(\vec{\omega} \times \vec{v}_{\text{rel}}) = -2(1)(36\hat{a}_y) = -72\hat{a}_y \,\text{N}.$$

(iii) Centrifugal Force: The Centrifugal force is given by $\vec{F}_{\text{centrifugal}} = -m\,\vec{\omega} \times (\vec{\omega} \times \vec{r})$. We have already calculated $\vec{\omega} \times \vec{r} = 12\hat{a}_z$. Now, calculate $\vec{\omega} \times (\vec{\omega} \times \vec{r})$:

$$\begin{aligned}
\vec{\omega} \times (\vec{\omega} \times \vec{r}) &= (3\hat{a}_x) \times (12\hat{a}_z) \\
&= (3 \times 12)(\hat{a}_x \times \hat{a}_z) \\
&= 36(-\hat{a}_y) \\
&= -36\hat{a}_y.
\end{aligned}$$

Finally, calculate the Centrifugal force:

$$\vec{F}_{\text{centrifugal}} = -m\,\vec{\omega} \times (\vec{\omega} \times \vec{r}) = -(1)(-36\hat{a}_y) = 36\hat{a}_y \,\text{N}.$$

Conclusion: The fictitious forces acting on the unit mass projectile in the rotating frame S' are:

- **Euler's Force**: $\vec{F}_{\text{Euler}} = \vec{0}$,
- **Coriolis Force**: $\vec{F}_{\text{Coriolis}} = -72\hat{a}_y \,\text{N}$,
- **Centrifugal Force**: $\vec{F}_{\text{centrifugal}} = 36\hat{a}_y \,\text{N}$.

35 A uniform solid sphere of radius R having moment of inertia I about its diameter is melted to form a uniform disc of thickness t and radius r. The moment of inertia of the disc about an axis passing through its edge and perpendicular to the plane is also equal to I. Show that the radius r of the disc is given by $r = \frac{2R}{\sqrt{15}}$.

Introduction: We are given a uniform solid sphere of radius R with moment of inertia I about its diameter. This sphere is melted and recast into a uniform disc of radius r and thickness t. The moment of inertia of the disc about an axis passing through its edge and perpendicular to the plane is also I. We are to find the radius r in terms of R and show that $r = \dfrac{2R}{\sqrt{15}}$. The total mass remains conserved during the transformation.

Solution:

Step 1: Moment of inertia of solid sphere about its diameter

Let the mass of the sphere be M. The moment of inertia of a uniform solid sphere about its diameter is:

$$I = \frac{2}{5}MR^2. \quad (1)$$

Step 2: Mass conservation

Let the density of the material be ρ.

Volume of the sphere:

$$V_{\text{sphere}} = \frac{4}{3}\pi R^3 \Rightarrow M = \rho \cdot \frac{4}{3}\pi R^3. \quad (2)$$

Volume of the disc:

$$V_{\text{disc}} = \pi r^2 t \Rightarrow M = \rho \cdot \pi r^2 t. \quad (3)$$

Equating (2) and (3):

$$\rho \cdot \frac{4}{3}\pi R^3 = \rho \cdot \pi r^2 t \Rightarrow \frac{4}{3}R^3 = r^2 t. \quad (4)$$

Step 3: Moment of inertia of the disc about an axis through its edge and perpendicular to the plane

The moment of inertia of a uniform disc of mass M and radius r about an axis perpendicular to its plane and passing through its edge is given by:

$$I = I_{\text{center}} + Mr^2 = \frac{1}{2}Mr^2 + Mr^2 = \frac{3}{2}Mr^2. \quad (5)$$

Equating this to the sphere's moment of inertia from (1):

$$\frac{3}{2}Mr^2 = \frac{2}{5}MR^2. \quad (6)$$

Cancel M on both sides:

$$\frac{3}{2}r^2 = \frac{2}{5}R^2 \Rightarrow r^2 = \frac{4}{15}R^2. \quad (7)$$

Taking square root:

$$r = \frac{2R}{\sqrt{15}}.$$

Conclusion: The radius r of the disc formed by melting a solid sphere of radius R and recasting it into a uniform disc such that the moment of inertia of the disc about an axis through its edge perpendicular to the plane equals that of the original sphere about its diameter is:

$$r = \frac{2R}{\sqrt{15}}.$$

This result follows from conservation of mass and standard expressions for the moment of inertia of a solid sphere and a disc.

The original UPSC questions was to show that the radius r of the disc is given by $r = 2R$. But in all mathematical certainty that was wrong or typographical error. If you feel otherwise, let me know at email abhksinhaphy@gmail.com

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36 What are Eulerian angles? A body with rotational symmetry about an axis is rotating under gravity about a point on the axis without friction. What quantities remain constant during the motion? Express them in terms of suitable Eulerian angles. Explain 'precession' and 'nutation' of such a body.

Introduction: Eulerian angles are a set of three angles that describe the orientation of a rigid body with respect to a fixed coordinate system. They are crucial in the study of the dynamics of rotating bodies. In this problem, we examine a rigid body (such as a symmetric top) with rotational symmetry about one axis, rotating under gravity about a fixed point on the axis of symmetry. We are to identify the conserved quantities during this motion, express them using Euler angles, and explain the phenomena of precession and nutation.

Solution:

Eulerian Angles:

Euler angles (ϕ, θ, ψ) describe the orientation of a rotating body relative to a fixed coordinate system through three successive rotations:

(i) ϕ (precession angle): rotation about the fixed z-axis.

(ii) θ (nutation angle): inclination of the body's symmetry axis with respect to the vertical (z-axis).

(iii) ψ (spin angle): rotation about the body's own symmetry axis.

These angles specify the transformation from the inertial frame to the body-fixed frame.

System Description:

Consider a symmetric top with moment of inertia $I_1 = I_2 \neq I_3$ about its principal axes. The body rotates about a fixed point (e.g., the tip) on its axis of symmetry in the presence of gravity. The center of mass lies on the symmetry axis at a distance l from the fixed point.

Let:

- M be the mass of the body,
- g be the acceleration due to gravity,
- I_1, I_3 be the moments of inertia (about perpendicular and symmetry axes, respectively),
- θ be the angle between the symmetry axis and the vertical.

Constants of Motion:

In the absence of friction and external torques at the fixed point, the following quantities are conserved:

(i) Total mechanical energy: The total energy includes rotational kinetic energy and gravitational potential energy:

$$E = T + U = \frac{1}{2}I_1(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 + Mgl\cos\theta. \quad (1)$$

(ii) Component of angular momentum along the vertical (inertial z-axis): The projection of the angular momentum on the vertical axis is conserved:

$$L_z = I_1\dot{\phi}\sin^2\theta + I_3(\dot{\psi} + \dot{\phi}\cos\theta)\cos\theta = \text{constant}. \quad (2)$$

(iii) Component of angular momentum along the symmetry axis: Since the body is symmetric and no torque acts along the symmetry axis,

$$L_3 = I_3(\dot{\psi} + \dot{\phi}\cos\theta) = \text{constant}. \quad (3)$$

Precession and Nutation:

Precession: It is the slow rotation of the symmetry axis of the top around the vertical axis (the z-axis). Mathematically, it corresponds to the time evolution of the angle $\phi(t)$:

$$\dot{\phi} = \text{precession angular velocity.}$$

This describes how the projection of the symmetry axis rotates in the horizontal plane.

Nutation: This is the oscillatory motion of the inclination angle $\theta(t)$. While precession corresponds to the ”sweeping” of the top's axis around the vertical, nutation represents the periodic ”nodding” motion of the symmetry axis due to changes in θ. Nutation occurs when the vertical component of angular momentum remains fixed but the inclination θ varies with time.

Summary of Dynamics:

- $\phi(t)$ increases steadily: **precession**.
- $\theta(t)$ oscillates: **nutation**.
- $\psi(t)$ evolves due to spin around the symmetry axis.

Conclusion: Eulerian angles (ϕ, θ, ψ) provide a complete description of the orientation of a rigid body in three dimensions. For a symmetric top rotating under gravity about a frictionless point:

(i) Total energy E is conserved,

(ii) Angular momentum component along vertical (L_z) is conserved,

(iii) Angular momentum along symmetry axis (L_3) is conserved.

Precession is the steady rotation of the symmetry axis about the vertical, and nutation is the oscillation of the inclination angle θ of the symmetry axis.

37 A rigid body is rotating about a fixed point with angular velocity $\vec{\omega}$. Assuming the coordinate axes coincide with the principal axes, if T is the kinetic energy and $\vec{G}$ is the external torque acting on the body, show that $\frac{dT}{dt} = \vec{G} \cdot \vec{\omega}$.

Introduction: We consider a rigid body rotating about a fixed point (typically the center of mass or a pivot point) under the influence of an external torque $\vec{G}$. The rotation is characterized by angular velocity $\vec{\omega}$. The coordinate system is aligned with the principal axes of the rigid body at the point of rotation. The objective is to prove that the time derivative of the rotational kinetic energy T is equal to the scalar product of the torque vector $\vec{G}$ with the angular velocity vector $\vec{\omega}$.

Solution:

Step 1: Rotational Kinetic Energy

In terms of the principal moments of inertia I_1, I_2, and I_3, and angular velocity components ω_1, ω_2, and ω_3 along the principal axes, the rotational kinetic energy is:

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2).$$

Step 2: Time Derivative of Kinetic Energy

Taking the time derivative of T:

$$\begin{aligned}\frac{dT}{dt} &= \frac{1}{2}\left(2I_1\omega_1\dot{\omega}_1 + 2I_2\omega_2\dot{\omega}_2 + 2I_3\omega_3\dot{\omega}_3\right) \\ &= I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3. \quad (1)\end{aligned}$$

Step 3: Euler's Equations of Motion

In a rotating frame aligned with the principal axes and fixed at the rotation point, the Euler equations for a rigid body are:

$$\begin{aligned}I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= G_1, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= G_2, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= G_3,\end{aligned}$$

where G_1, G_2, G_3 are the components of the external torque $\vec{G}$ in the body frame.

Rewriting the angular acceleration terms from above:

$$\begin{aligned}I_1\dot{\omega}_1 &= G_1 - (I_3 - I_2)\omega_2\omega_3, \\ I_2\dot{\omega}_2 &= G_2 - (I_1 - I_3)\omega_3\omega_1, \\ I_3\dot{\omega}_3 &= G_3 - (I_2 - I_1)\omega_1\omega_2.\end{aligned}$$

Substitute these into equation (1):

$$\begin{aligned}
\frac{dT}{dt} &= \omega_1\left[G_1-(I_3-I_2)\omega_2\omega_3\right]+\omega_2\left[G_2-(I_1-I_3)\omega_3\omega_1\right] \\
&\quad +\omega_3\left[G_3-(I_2-I_1)\omega_1\omega_2\right] \\
&= G_1\omega_1+G_2\omega_2+G_3\omega_3 \\
&\quad -(I_3-I_2)\omega_1\omega_2\omega_3-(I_1-I_3)\omega_2\omega_3\omega_1-(I_2-I_1)\omega_3\omega_1\omega_2.
\end{aligned}$$

Now observe that all the triple-product terms cancel pairwise:

$$-(I_3-I_2)\omega_1\omega_2\omega_3-(I_1-I_3)\omega_1\omega_2\omega_3-(I_2-I_1)\omega_1\omega_2\omega_3=0.$$

Hence:

$$\frac{dT}{dt}=G_1\omega_1+G_2\omega_2+G_3\omega_3=\vec{G}\cdot\vec{\omega}.$$

Conclusion: For a rigid body rotating about a fixed point with angular velocity $\vec{\omega}$, the time rate of change of the rotational kinetic energy T is equal to the scalar product of the external torque $\vec{G}$ and angular velocity:

$$\frac{dT}{dt}=\vec{G}\cdot\vec{\omega}.$$

This result is valid when the coordinate system is aligned with the principal axes of the body and encapsulates the work-energy theorem in rotational dynamics.

38 Determine the number of degrees of freedom for a rigid body: (i) moving freely in 3D space, (ii) having one point fixed, (iii) having two points fixed.

Introduction: The degrees of freedom (DOF) of a mechanical system represent the number of independent parameters required to completely describe its configuration. For a rigid body in three-dimensional space, these degrees arise from both translation and rotation. We analyze the number of DOF for three distinct cases of rigid body motion, guided by physical constraints and symmetry considerations.

Solution:

(i) Rigid body moving freely in 3D space:

A general rigid body in three-dimensional space has:

- 3 translational degrees of freedom: motion along x, y, and z axes,
- 3 rotational degrees of freedom: rotations about x, y, and z axes (often described by Euler angles: ϕ, θ, ψ).

Hence, the total number of degrees of freedom is:

$$\text{DOF} = 3 + 3 = 6.$$

(ii) Rigid body with one point fixed:

If one point of the rigid body is fixed (e.g., a top pivoted at a point), it cannot translate. However, it can still rotate about any of the three orthogonal axes through the fixed point. Therefore:

$$\text{DOF} = 3 \quad \text{(rotational)}.$$

(iii) Rigid body with two points fixed:

Fixing two points of the body (not coinciding) constrains translation and all but one rotation. The body can only rotate about the axis passing through these two fixed points. This leaves only one independent rotational motion:

$$\text{DOF} = 1.$$

Conclusion:

(i) A rigid body moving freely in 3D space has **6 degrees of freedom**.

(ii) A rigid body with one point fixed has **3 degrees of freedom** (rotational).

(iii) A rigid body with two points fixed has **1 degree of freedom** (rotation about the axis connecting the fixed points).

These results reflect the progressive constraints imposed on the rigid body by fixing points in space.

39 Calculate the moment of inertia of a solid cone of mass M, height h, vertical half-angle α, and radius of its base R, about an axis passing through its vertex and parallel to its base.

Introduction: We are to calculate the moment of inertia of a solid cone of mass M, height h, base radius R, and vertical half-angle α, about an axis passing through its vertex and parallel to the base plane (i.e., perpendicular to the cone's central axis). The cone is uniform and solid. We'll use cylindrical coordinates with careful attention to the geometric constraints.

Solution:

Step 1: Coordinate System Setup

Let the vertex of the cone be at the origin and the cone's central axis aligned along the positive z-axis. The base of the cone lies in the plane $z = h$. We calculate the moment of inertia about the x-axis, which passes through the vertex and is parallel to the base plane.

For rotation about the x-axis, the perpendicular distance from any point (x, y, z) to the rotation axis is:

$$r_\perp^2 = y^2 + z^2$$

This is the squared distance measured in the yz-plane, perpendicular to the x-axis.

Step 2: Cone Geometry and Density

The cone's surface is described by the relationship between the radial distance from the z-axis and the height:

$$r = \frac{R}{h}z = z\tan\alpha, \quad \text{where } \tan\alpha = \frac{R}{h}$$

At height z, the maximum radial extent is $r_{\text{max}}(z) = \frac{R}{h}z$.

Volume of the cone:

$$V = \frac{1}{3}\pi R^2 h$$

Mass density (uniform):

$$\rho = \frac{M}{V} = \frac{3M}{\pi R^2 h}$$

Step 3: Integration Setup in Cylindrical Coordinates

Using cylindrical coordinates (r, θ, z):

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z, \quad dV = r\,dr\,d\theta\,dz$$

Integration bounds:

- θ: from 0 to 2π (full rotation)
- z: from 0 to h (vertex to base)
- r: from 0 to $\frac{R}{h}z$ (center to cone boundary at height z)

Step 4: Moment of Inertia Calculation

The moment of inertia about the x-axis is:

$$I_x = \iiint_V \rho(y^2 + z^2)\, dV$$

Substituting $y^2 = r^2 \sin^2 \theta$:

$$I_x = \rho \int_0^{2\pi} \int_0^h \int_0^{z \tan \alpha} (r^2 \sin^2 \theta + z^2) \cdot r\, dr\, d\theta\, dz$$

Separating the integral:

$$I_x = \rho \int_0^{2\pi} \int_0^h \left[\int_0^{z \tan \alpha} r^3 \sin^2 \theta\, dr + z^2 \int_0^{z \tan \alpha} r\, dr \right] d\theta\, dz$$

Step 5: Inner Integrations

Computing the r-integrals:

$$\int_0^{z \tan \alpha} r^3\, dr = \frac{1}{4}(z \tan \alpha)^4, \quad \int_0^{z \tan \alpha} r\, dr = \frac{1}{2}(z \tan \alpha)^2$$

Substituting:

$$I_x = \rho \int_0^{2\pi} \int_0^h \left[\frac{1}{4} z^4 \tan^4 \alpha \sin^2 \theta + \frac{1}{2} z^4 \tan^2 \alpha \right] d\theta\, dz$$

Step 6: Angular Integration

Computing the θ-integrals:

$$\int_0^{2\pi} \sin^2 \theta\, d\theta = \pi, \quad \int_0^{2\pi} d\theta = 2\pi$$

Therefore:

$$I_x = \rho \int_0^h z^4 \left[\frac{\pi}{4} \tan^4 \alpha + \pi \tan^2 \alpha \right] dz = \rho\pi \int_0^h z^4 \tan^2 \alpha \left(\frac{1}{4} \tan^2 \alpha + 1 \right) dz$$

Step 7: Final Integration and Substitution

Computing the z-integral:

$$\int_0^h z^4\, dz = \frac{h^5}{5}$$

Thus:

$$I_x = \rho\pi \tan^2 \alpha \left(\frac{1}{4} \tan^2 \alpha + 1 \right) \frac{h^5}{5}$$

Substituting $\rho = \frac{3M}{\pi R^2 h}$ and $\tan\alpha = \frac{R}{h}$:

$$\begin{aligned} I_x &= \frac{3M}{\pi R^2 h} \cdot \pi \cdot \frac{R^2}{h^2} \left(\frac{1}{4} \cdot \frac{R^2}{h^2} + 1 \right) \cdot \frac{h^5}{5} \\ &= \frac{3M}{5} \cdot \frac{R^2 h^2}{h^2} \left(\frac{R^2}{4h^2} + 1 \right) \\ &= \frac{3M}{5} R^2 \left(\frac{R^2}{4h^2} + 1 \right) \end{aligned}$$

Conclusion: The moment of inertia of a solid cone of mass M, base radius R, and height h about an axis passing through its vertex and parallel to the base plane is:

$$I = \frac{3M}{5} R^2 \left(1 + \frac{R^2}{4h^2} \right)$$

This result accounts for the geometric distribution of mass relative to the specified rotation axis, with the first term representing the contribution from the radial distribution and the second term from the axial distribution of mass.

40 Show that the kinetic energy and angular momentum of torque-free motion of a rigid body are constant.

Introduction: We are to demonstrate that in the absence of external torques (i.e., under torque-free motion), both the kinetic energy and the angular momentum of a rigid body remain constant over time. This is a fundamental result in rigid body dynamics and follows from the conservation laws applied to isolated systems. We will work in the body-fixed frame aligned with the principal axes of inertia.

Solution:

Let the principal moments of inertia of the rigid body be I_1, I_2, and I_3, and let the components of the angular velocity vector $\vec{\omega}$ in the body-fixed frame be ω_1, ω_2, and ω_3.

(i) Angular Momentum Conservation

In the absence of external torque, Newton's second law for rotation gives:

$$\frac{d\vec{L}}{dt}_{\text{space}} = \vec{G}_{\text{ext}} = \vec{0}.$$

Using the transport theorem:

$$\frac{d\vec{L}}{dt}_{\text{space}} = \frac{d\vec{L}}{dt}_{\text{body}} + \vec{\omega} \times \vec{L}.$$

So, in the torque-free case:

$$\frac{d\vec{L}}{dt}_{\text{body}} = -\vec{\omega} \times \vec{L}. \quad (1)$$

In the body frame aligned with principal axes, the angular momentum is:

$$\vec{L} = I_1\omega_1\hat{e}_1 + I_2\omega_2\hat{e}_2 + I_3\omega_3\hat{e}_3.$$

Euler's equations for torque-free motion are:

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0, \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0, \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0. \quad (2) \end{aligned}$$

These equations govern the evolution of $\omega_i(t)$ in torque-free motion.

To show $\vec{L}$ is constant in the inertial frame, note from equation (1) that its time derivative in the body frame is a cross product with $\vec{\omega}$, which means its magnitude and direction in inertial space remain constant.

(ii) Kinetic Energy Conservation

Rotational kinetic energy of the rigid body is:

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2).$$

Take time derivative:

$$\frac{dT}{dt} = I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3.$$

Substitute $\dot{\omega}_i$ from Euler's equations (2):

$$\begin{aligned} I_1\omega_1\dot{\omega}_1 &= -(I_3 - I_2)\omega_1\omega_2\omega_3, \\ I_2\omega_2\dot{\omega}_2 &= -(I_1 - I_3)\omega_1\omega_2\omega_3, \\ I_3\omega_3\dot{\omega}_3 &= -(I_2 - I_1)\omega_1\omega_2\omega_3. \end{aligned}$$

So,

$$\frac{dT}{dt} = -\omega_1\omega_2\omega_3\left[(I_3 - I_2) + (I_1 - I_3) + (I_2 - I_1)\right] = 0.$$

Therefore:

$$\frac{dT}{dt} = 0.$$

Conclusion: In the absence of external torque:

(i) The angular momentum vector $\vec{L}$ remains constant in inertial space,

(ii) The kinetic energy T of the rigid body remains constant.

These results follow directly from Euler's equations and reflect conservation of angular momentum and mechanical energy in torque-free motion.