UPSC PHYSICS PYQ SOLUTION
Mechanics - Part 5

Contents

41 If $I^{'}$ and I be the moments of inertia of a body about an axis passing through an arbitrary origin and about a parallel axis through the center of mass, respectively, show that $I^{'} = MR^2 + I$, where $\vec{R}$ is the position vector of the center of mass with respect to the arbitrary origin and M is the mass of the body.

Introduction: The problem requires us to derive the relation between the moment of inertia I' of a rigid body about an arbitrary axis and the moment of inertia I about a parallel axis through its center of mass. We are given the total mass M of the body and the position vector $\vec{R}$ of the center of mass relative to the arbitrary origin. The goal is to prove the parallel axis theorem:

$$I' = MR^2 + I$$

We will approach this by considering the definition of moment of inertia and applying coordinate transformation.

Solution: Consider a rigid body consisting of mass elements m_i at position vectors $\vec{r}_i$ with respect to an arbitrary origin O. Let the two parallel axes be perpendicular to the plane of motion.

The position vector of the center of mass is defined as:

$$\vec{R} = \frac{1}{M}\sum_i m_i\vec{r}_i$$

where $M = \sum_i m_i$ is the total mass.

Define $\vec{r}_i' = \vec{r}_i - \vec{R}$ as the position vector of the ith mass element relative to the center of mass.

The moment of inertia I' about the axis through the arbitrary origin is:

$$I' = \sum_i m_i|\vec{r}_i|^2$$

The moment of inertia I about the parallel axis through the center of mass is:

$$I = \sum_i m_i|\vec{r}_i'|^2$$

Now, substituting $\vec{r}_i = \vec{R} + \vec{r}_i'$ into the expression for I':

$$I' = \sum_i m_i|\vec{R} + \vec{r}_i'|^2$$

Expanding the square of the vector sum:

$$I' = \sum_i m_i(\vec{R} + \vec{r}_i') \cdot (\vec{R} + \vec{r}_i')$$

$$I' = \sum_i m_i(|\vec{R}|^2 + 2\vec{R}\cdot\vec{r}_i' + |\vec{r}_i'|^2)$$

Distributing the summation:

$$I' = \sum_i m_i|\vec{R}|^2 + 2\vec{R}\cdot\sum_i m_i\vec{r}_i' + \sum_i m_i|\vec{r}_i'|^2$$

Evaluating each term:

(i) $|\vec{R}|^2 = R^2$ is constant, so $\sum_i m_i R^2 = MR^2$

(ii) $\sum_i m_i\vec{r}_i' = \sum_i m_i(\vec{r}_i - \vec{R}) = \sum_i m_i\vec{r}_i - M\vec{R} = M\vec{R} - M\vec{R} = 0$ (by definition of center of mass)

(iii) $\sum_i m_i|\vec{r}_i'|^2 = I$ (moment of inertia about center of mass)

Therefore:

$$I' = MR^2 + 2\vec{R}\cdot\vec{0} + I = MR^2 + I$$

Conclusion: We have proven that the moment of inertia I' of a rigid body about an arbitrary axis is related to the moment of inertia I about a parallel axis through the center of mass by:

$$I' = MR^2 + I$$

This fundamental result is known as the **parallel axis theorem** (or Steiner's theorem), which is essential in rotational dynamics for calculating moments of inertia about different axes.

42 Consider a rigid body rotating about an axis passing through a fixed point in the body with an angular velocity $\vec{\omega}$. Determine the kinetic energy of such a rotating body in a coordinate system of principal axes. If the Earth suddenly stops rotating, what will happen to the rotational kinetic energy? Comment in detail.

Introduction: We are to determine the kinetic energy of a rigid body rotating with angular velocity $\vec{\omega}$ about a fixed point using a coordinate system aligned with the principal axes of inertia. Additionally, we must discuss the physical implications if the Earth, a rotating rigid body, were to suddenly stop spinning.

Let the body have moment of inertia tensor **I** and mass distribution such that the rotation is analyzed with respect to principal axes at the fixed point. The goal is to express kinetic energy in terms of the angular velocity components and principal moments of inertia.

Solution:

Part 1: Kinetic Energy in Principal Axes

The kinetic energy T of a rotating rigid body about a fixed point is given by:

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{L}$$

where $\vec{L} = \mathbf{I}\vec{\omega}$ is the angular momentum.

In a principal axis coordinate system, the inertia tensor **I** is diagonal:

$$\mathbf{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Let the angular velocity vector be:

$$\vec{\omega} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{k}$$

Then the angular momentum vector is:

$$\vec{L} = I_1\omega_1 \hat{i} + I_2\omega_2 \hat{j} + I_3\omega_3 \hat{k}$$

The kinetic energy becomes:

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}(\omega_1, \omega_2, \omega_3) \cdot (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

This expression shows that in a principal axis system, the rotational kinetic energy is a simple sum of the energy contributions along each principal direction.

Part 2: Earth's Rotational Kinetic Energy

The Earth rotates about its polar axis (approximately a principal axis) with: - Moment of inertia: $I_E \approx 8.0 \times 10^{37}$ kg□m^2 - Angular velocity: $\omega_E = \frac{2\pi}{24\times 3600} \approx 7.3 \times 10^{-5}$ rad/s

The Earth's rotational kinetic energy is:

$$T_E = \frac{1}{2} I_E \omega_E^2 \approx \frac{1}{2} \times 8.0 \times 10^{37} \times (7.3 \times 10^{-5})^2 \approx 2.1 \times 10^{29} \text{ J}$$

Physical Consequences if Earth Stops Rotating:

If the Earth suddenly stops rotating ($\omega_E \to 0$), the rotational kinetic energy $T_E \to 0$. This enormous energy ($\sim 10^{29}$ J) must be converted to other forms:

1. Atmospheric and Oceanic Inertia: The atmosphere and oceans would continue moving eastward at speeds up to 1670 km/h (at the equator) due to inertia. This would create:

- Supersonic winds causing complete atmospheric redistribution
- Massive tsunamis as oceans slam into continental barriers
- Complete destruction of all surface structures

2. Internal Heating: The kinetic energy conversion would cause:

- Extreme frictional heating throughout the Earth's crust and mantle
- Potential melting of significant portions of the Earth's surface
- Massive volcanic activity and crustal deformation

3. Gravitational and Tidal Effects:

- Earth's oblate shape (due to rotation) would begin to change, causing massive earthquakes
- Tidal patterns would be completely disrupted
- The length of day would become equal to the orbital period (1 year)

4. Conservation Laws: From a physics standpoint, such an event would require:

- An external torque of magnitude $|\vec{\tau}| = \frac{d\vec{L}}{dt} = I_E \omega_E / \Delta t$
- For a sudden stop ($\Delta t \to 0$), this torque approaches infinity
- This violates the principle that no infinite forces exist in nature

5. Reference Frame Considerations: The "sudden stop" is relative to the inertial reference frame. From the perspective of energy conservation, the 2.1×10^{29} J would manifest as:

- Kinetic energy of atmospheric motion: $\sim 10^{27}$ J
- Seismic energy from crustal readjustment: $\sim 10^{26}$ J
- Thermal energy from friction: $\sim 10^{29}$ J (the majority)

Conclusion: The kinetic energy of a rigid body rotating about a fixed point in a principal axis system is:

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

If the Earth were to stop rotating suddenly, its enormous rotational kinetic energy ($\sim 2.1 \times 10^{29}$ J) would be converted into catastrophic forms of energy including atmospheric motion, seismic activity, and thermal heating. Such an event is physically impossible without infinite external torque, demonstrating both the conservation of angular momentum and the immense scale of planetary rotational energy.

43 A body turns about a fixed point. Show that the angle between its angular velocity vector and its angular momentum vector about the fixed point is always acute.

Introduction: We are to show that for a rigid body rotating about a fixed point, the angle between its angular velocity vector $\vec{\omega}$ and its angular momentum vector $\vec{L}$ about the fixed point is always acute. This entails demonstrating that the scalar product $\vec{\omega} \cdot \vec{L}$ is always positive, which implies that the angle θ between them satisfies $0 \leq \theta < \frac{\pi}{2}$.

Solution:

Consider a rigid body rotating about a fixed point O with angular velocity $\vec{\omega}$. Let the body consist of mass elements m_i at position vectors $\vec{r}_i$ from point O.

The velocity of the ith mass element is:

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

The angular momentum about point O is:

$$\vec{L} = \sum_i m_i(\vec{r}_i \times \vec{v}_i) = \sum_i m_i\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

Using the vector triple product identity $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$:

$$\vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \vec{\omega}(\vec{r}_i \cdot \vec{r}_i) - \vec{r}_i(\vec{r}_i \cdot \vec{\omega}) = \vec{\omega}r_i^2 - \vec{r}_i(\vec{r}_i \cdot \vec{\omega})$$

Therefore:

$$\vec{L} = \sum_i m_i[\vec{\omega}r_i^2 - \vec{r}_i(\vec{r}_i \cdot \vec{\omega})]$$

This can be written in tensor form as:

$$\vec{L} = \mathbf{I}\vec{\omega}$$

where $\mathbf{I}$ is the inertia tensor with components:

$$I_{jk} = \sum_i m_i(r_i^2\delta_{jk} - r_{ij}r_{ik})$$

Now, the kinetic energy of the rotating body is:

$$T = \frac{1}{2}\sum_i m_i v_i^2 = \frac{1}{2}\sum_i m_i|\vec{\omega} \times \vec{r}_i|^2$$

Using the identity $|\vec{a} \times \vec{b}|^2 = a^2b^2 - (\vec{a} \cdot \vec{b})^2$:

$$|\vec{\omega} \times \vec{r}_i|^2 = \omega^2 r_i^2 - (\vec{\omega} \cdot \vec{r}_i)^2$$

Therefore:

$$T = \frac{1}{2}\sum_i m_i[\omega^2 r_i^2 - (\vec{\omega} \cdot \vec{r}_i)^2]$$

Alternatively, we can express kinetic energy as:

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{L} = \frac{1}{2}\vec{\omega} \cdot (\mathbf{I}\vec{\omega})$$

To prove this relationship, we compute:

$$\vec{\omega} \cdot \vec{L} = \vec{\omega} \cdot \sum_i m_i[\vec{\omega} r_i^2 - \vec{r}_i(\vec{r}_i \cdot \vec{\omega})]$$

$$= \sum_i m_i[\omega^2 r_i^2 - (\vec{\omega} \cdot \vec{r}_i)^2] = 2T$$

Since the inertia tensor is positive definite (a fundamental property of mass distributions), we have:

$$\vec{\omega} \cdot (\mathbf{I}\vec{\omega}) > 0 \quad \text{for all } \vec{\omega} \neq \vec{0}$$

This can be proven by noting that for any real vector $\vec{\omega}$:

$$\vec{\omega} \cdot (\mathbf{I}\vec{\omega}) = \sum_i m_i[\omega^2 r_i^2 - (\vec{\omega} \cdot \vec{r}_i)^2]$$

By the Cauchy-Schwarz inequality, $(\vec{\omega} \cdot \vec{r}_i)^2 \leq \omega^2 r_i^2$, with equality only when $\vec{r}_i$ is parallel to $\vec{\omega}$. Since the body is three-dimensional (not all mass elements lie on a single line through O), there exist mass elements for which $\vec{r}_i$ is not parallel to $\vec{\omega}$, ensuring:

$$\sum_i m_i[\omega^2 r_i^2 - (\vec{\omega} \cdot \vec{r}_i)^2] > 0$$

Therefore:

$$T = \frac{1}{2}\vec{\omega} \cdot \vec{L} > 0 \quad \text{for } \vec{\omega} \neq \vec{0}$$

This implies:

$$\vec{\omega} \cdot \vec{L} > 0$$

Since $\vec{\omega} \cdot \vec{L} = |\vec{\omega}||\vec{L}| \cos\theta > 0$, and both $|\vec{\omega}| > 0$ and $|\vec{L}| > 0$ for non-zero rotation, we must have:

$$\cos\theta > 0 \Rightarrow 0 \leq \theta < \frac{\pi}{2}$$

Conclusion: The angle between the angular velocity vector $\vec{\omega}$ and the angular momentum vector $\vec{L}$ of a rigid body rotating about a fixed point is always acute. This follows from the positive definiteness of the inertia tensor, which ensures that $\vec{\omega} \cdot \vec{L} = 2T > 0$ for any non-zero rotation, where T is the kinetic energy.

44 How does one obtain the angular velocity of the Earth about the North Pole with respect to a fixed star as $7.292 \times 10^{-5}\, s^{-1}$? Explain your method of calculating the above value.

Introduction: The question involves calculating the angular velocity of the Earth as it rotates about its axis (which passes through the North and South Poles) with respect to a fixed star. This angular velocity corresponds to the sidereal rotation period of the Earth — the time it takes for the Earth to complete one full rotation relative to the fixed stars, rather than the Sun.

Solution:

The angular velocity ω of any rotating object is given by:

$$\omega = \frac{2\pi}{T}$$

where T is the period of rotation in seconds.

For the Earth, the sidereal day (rotation period with respect to fixed stars) is:

$$T = 23\,\mathrm{h}\,56\,\mathrm{min}\,4.091\,\mathrm{s}$$

Convert this to seconds:

$$\begin{aligned} T &= 23 \times 3600 + 56 \times 60 + 4.091 \\ &= 82800 + 3360 + 4.091 \\ &= 86164.091\,\mathrm{s} \end{aligned}$$

Now compute the angular velocity:

$$\omega = \frac{2\pi}{86164.091}\,\mathrm{s}^{-1}$$

Evaluating:

$$\omega \approx \frac{6.283185}{86164.091} \approx 7.292 \times 10^{-5}\,\mathrm{s}^{-1}$$

Conclusion: The Earth's angular velocity about the North Pole with respect to a fixed star is obtained by dividing 2π radians by the sidereal day duration in seconds, yielding:

$$\omega \approx 7.292 \times 10^{-5}\,\mathrm{s}^{-1}$$

This represents the Earth's uniform angular speed as it rotates once per sidereal day relative to the fixed stars.

45 Show that the moment of inertia of a circular disc of mass M and radius R about an axis passing through its center and perpendicular to its plane is $\frac{1}{2}MR^2$.

Introduction: The problem is to derive the moment of inertia of a uniform circular disc about an axis that passes through its center and is perpendicular to its plane. We consider a planar disc of total mass M and radius R, with mass uniformly distributed. The calculation uses integration over the area of the disc to sum contributions of elemental mass at varying distances from the axis.

Solution:

Consider a circular disc of radius R and total mass M, with uniform surface mass density. The axis of rotation is through the center and perpendicular to the plane of the disc.

Let σ be the mass per unit area:

$$\sigma = \frac{M}{\pi R^2}$$

We use polar coordinates (r, θ) to integrate over the area of the disc. Consider an elemental ring of radius r and thickness dr.

The area of the ring is:

$$dA = 2\pi r\,dr$$

The mass of the ring is:

$$dm = \sigma\,dA = \sigma \cdot 2\pi r\,dr$$

Each mass element in the ring is at a distance r from the axis of rotation, so its moment of inertia is:

$$dI = r^2\,dm = r^2 \cdot \sigma \cdot 2\pi r\,dr = 2\pi\sigma r^3\,dr$$

Integrate dI from $r = 0$ to $r = R$:

$$I = \int_0^R 2\pi\sigma r^3\,dr = 2\pi\sigma \int_0^R r^3\,dr = 2\pi\sigma \left[\frac{r^4}{4}\right]_0^R = \frac{1}{2}\pi\sigma R^4$$

Substitute $\sigma = \frac{M}{\pi R^2}$:

$$I = \frac{1}{2}\pi \cdot \frac{M}{\pi R^2} \cdot R^4 = \frac{1}{2}MR^2$$

Conclusion: The moment of inertia of a uniform circular disc of mass M and radius R about an axis through its center and perpendicular to its plane is:

$$I = \frac{1}{2}MR^2$$

This result is fundamental in planar rotational dynamics and is widely used in mechanical and physical applications.

46 Four solid spheres A, B, C, and D, each of mass m and radius a, are placed with their centers on the four corners of a square of side b. Calculate the moment of inertia of the system about one side of the square. Also, calculate the moment of inertia of the system about a diagonal of the square.

Introduction: We are given four identical solid spheres of mass m and radius a placed at the corners of a square of side b. The task is to calculate the moment of inertia (MI) of this system:

(a) About one side of the square (say, the side through centers of spheres A and B),

(b) About a diagonal of the square (say, the diagonal through spheres A and C).

Each sphere's moment of inertia includes both the moment about its own center and the additional term due to its displacement from the axis, using the parallel axis theorem.

Solution:

Let us establish a coordinate system and label the square's corners as:

- A at origin: (0,0)
- B: (b,0)
- C: (b,b)
- D: (0,b)

The moment of inertia of a solid sphere about any axis through its center is:

$$I_{\text{sphere,center}} = \frac{2}{5}ma^2$$

For any sphere displaced from the rotation axis, we apply the parallel axis theorem:

$$I_{\text{total}} = I_{\text{center}} + md^2$$

where d is the perpendicular distance from the sphere's center to the rotation axis.

Part (a): Moment of Inertia about side AB

The rotation axis is along the x-axis (the line segment AB).

Distances from the rotation axis:

- Sphere A at (0,0): $d_A = 0$ (center lies on the axis)
- Sphere B at (b,0): $d_B = 0$ (center lies on the axis)
- Sphere C at (b,b): $d_C = b$ (perpendicular distance from x-axis)
- Sphere D at (0,b): $d_D = b$ (perpendicular distance from x-axis)

Individual moments of inertia:

- $I_A = \frac{2}{5}ma^2 + m(0)^2 = \frac{2}{5}ma^2$
- $I_B = \frac{2}{5}ma^2 + m(0)^2 = \frac{2}{5}ma^2$
- $I_C = \frac{2}{5}ma^2 + mb^2$
- $I_D = \frac{2}{5}ma^2 + mb^2$

Total moment of inertia about side AB:

$$\begin{aligned} I_{\text{AB}} &= I_A + I_B + I_C + I_D && (1) \\ &= 2\left(\frac{2}{5}ma^2\right) + 2\left(\frac{2}{5}ma^2 + mb^2\right) && (2) \\ &= \frac{4}{5}ma^2 + \frac{4}{5}ma^2 + 2mb^2 && (3) \\ &= \frac{8}{5}ma^2 + 2mb^2 && (4) \end{aligned}$$

Part (b): Moment of Inertia about diagonal AC

The rotation axis is along the diagonal AC, which lies along the line $y = x$ from (0,0) to (b,b).

The perpendicular distance from a point (x_0, y_0) to the line $y = x$ is:

$$d = \frac{|x_0 - y_0|}{\sqrt{1^2 + (-1)^2}} = \frac{|x_0 - y_0|}{\sqrt{2}}$$

Distances from the rotation axis:

- Sphere A at (0,0): $d_A = \frac{|0-0|}{\sqrt{2}} = 0$ (lies on diagonal)
- Sphere B at (b,0): $d_B = \frac{|b-0|}{\sqrt{2}} = \frac{b}{\sqrt{2}}$
- Sphere C at (b,b): $d_C = \frac{|b-b|}{\sqrt{2}} = 0$ (lies on diagonal)
- Sphere D at (0,b): $d_D = \frac{|0-b|}{\sqrt{2}} = \frac{b}{\sqrt{2}}$

Individual moments of inertia:

- $I_A = \frac{2}{5}ma^2 + m(0)^2 = \frac{2}{5}ma^2$
- $I_B = \frac{2}{5}ma^2 + m\left(\frac{b}{\sqrt{2}}\right)^2 = \frac{2}{5}ma^2 + \frac{mb^2}{2}$
- $I_C = \frac{2}{5}ma^2 + m(0)^2 = \frac{2}{5}ma^2$
- $I_D = \frac{2}{5}ma^2 + m\left(\frac{b}{\sqrt{2}}\right)^2 = \frac{2}{5}ma^2 + \frac{mb^2}{2}$

Total moment of inertia about diagonal AC:

$$\begin{aligned}
I_{\text{AC}} &= I_A + I_B + I_C + I_D &(5)\\
&= 2\left(\frac{2}{5}ma^2\right) + 2\left(\frac{2}{5}ma^2 + \frac{mb^2}{2}\right) &(6)\\
&= \frac{4}{5}ma^2 + \frac{4}{5}ma^2 + mb^2 &(7)\\
&= \frac{8}{5}ma^2 + mb^2 &(8)
\end{aligned}$$

Conclusion: The moments of inertia of the system of four identical solid spheres placed at the corners of a square of side b are:

- About one side of the square:

$$I_{\text{side}} = \frac{8}{5}ma^2 + 2mb^2$$

- About a diagonal of the square:

$$I_{\text{diagonal}} = \frac{8}{5}ma^2 + mb^2$$

Note that the moment of inertia about the diagonal is smaller than that about the side, which is expected since the average distance of the spheres from the diagonal is less than from the side.

47 Define moment of inertia and explain its physical significance. Calculate the moment of inertia of an annular ring about an axis passing through its center and perpendicular to its plane.

Introduction: This problem consists of two parts. First, we are to define the moment of inertia (MI) and explain its physical meaning in the context of rotational dynamics. Second, we must calculate the MI of an annular ring (a flat ring with inner and outer radii) about an axis passing through its center and perpendicular to its plane.

Solution:

Definition and Physical Significance:

The **moment of inertia** I of a rigid body about a given axis is a scalar measure of the body's resistance to angular acceleration about that axis. It is defined as:

$$I = \sum_i m_i r_i^2$$

for discrete masses, or

$$I = \int r^2 \, dm$$

for continuous mass distributions, where r is the perpendicular distance of a mass element dm from the axis of rotation.

Physical significance:

- It plays a role analogous to mass in linear motion, appearing in the rotational analog of Newton's second law:
$$\tau = I\alpha$$
where τ is torque and α is angular acceleration.
- It determines the rotational kinetic energy:
$$T = \frac{1}{2}I\omega^2$$
where ω is the angular velocity.
- A larger moment of inertia implies more torque is needed to achieve the same angular acceleration.

Moment of Inertia of an Annular Ring:

Let the annular ring have:

- Inner radius: R_1
- Outer radius: R_2
- Total mass: M

Assume uniform surface mass density. We compute the MI about an axis perpendicular to its plane and through its center.

Let dm be the mass of an infinitesimal ring of radius r and thickness dr.

The area of the infinitesimal ring is:

$$dA = 2\pi r\, dr$$

Total area of the annular ring:

$$A = \pi(R_2^2 - R_1^2)$$

Surface mass density:

$$\sigma = \frac{M}{\pi(R_2^2 - R_1^2)}$$

Mass of infinitesimal ring:

$$dm = \sigma \cdot dA = \frac{M}{\pi(R_2^2 - R_1^2)} \cdot 2\pi r\, dr = \frac{2Mr\, dr}{R_2^2 - R_1^2}$$

Moment of inertia of this ring:

$$dI = r^2\, dm = \frac{2Mr^3\, dr}{R_2^2 - R_1^2}$$

Total moment of inertia:

$$\begin{aligned}
I &= \int_{R_1}^{R_2} \frac{2Mr^3\, dr}{R_2^2 - R_1^2} \\
&= \frac{2M}{R_2^2 - R_1^2} \int_{R_1}^{R_2} r^3\, dr \\
&= \frac{2M}{R_2^2 - R_1^2} \left[\frac{r^4}{4}\right]_{R_1}^{R_2} \\
&= \frac{2M}{R_2^2 - R_1^2} \cdot \frac{1}{4}(R_2^4 - R_1^4) \\
&= \frac{M}{2} \cdot \frac{R_2^4 - R_1^4}{R_2^2 - R_1^2}
\end{aligned}$$

Using the identity $a^2 - b^2 = (a-b)(a+b)$, we simplify:

$$I = \frac{M}{2}(R_1^2 + R_2^2)$$

Conclusion: The moment of inertia of an annular ring of mass M, inner radius R_1, and outer radius R_2, about an axis perpendicular to its plane and through its center, is:

$$I = \frac{M}{2}(R_1^2 + R_2^2)$$

This result generalizes the familiar $I = MR^2$ for a thin ring, which is recovered when $R_1 = R_2 = R$.

48 (i) Find the moments of inertia of a rigid diatomic molecule about different symmetry axes through the center of mass.

Introduction: A rigid diatomic molecule consists of two atoms of masses m_1 and m_2, separated by a fixed distance r. We need to compute the moments of inertia about all principal axes passing through the center of mass (COM). Due to the linear geometry, there are three principal axes with only two distinct moment of inertia values.

Setup: Let the molecular axis be along the z-direction. The center of mass divides the internuclear distance such that:

$$r_1 = \frac{m_2 r}{m_1 + m_2}, \quad r_2 = \frac{m_1 r}{m_1 + m_2}$$

where r_1 and r_2 are the distances from the COM to atoms 1 and 2, respectively.

Solution:

(a) Moment of inertia about the molecular axis (z-axis): Since both atoms lie on the z-axis, their perpendicular distances from this axis are zero:

$$I_z = I_{\parallel} = m_1 \cdot 0^2 + m_2 \cdot 0^2 = 0$$

(b) Moment of inertia about axes perpendicular to the molecular axis: For rotation about the x-axis (perpendicular to the molecular bond):

$$I_x = m_1 r_1^2 + m_2 r_2^2$$

Substituting the COM distances:

$$\begin{aligned} I_x &= m_1 \left(\frac{m_2 r}{m_1 + m_2}\right)^2 + m_2 \left(\frac{m_1 r}{m_1 + m_2}\right)^2 \\ &= \frac{m_1 m_2^2 r^2 + m_2 m_1^2 r^2}{(m_1 + m_2)^2} = \frac{m_1 m_2 r^2 (m_1 + m_2)}{(m_1 + m_2)^2} \\ I_x &= \frac{m_1 m_2 r^2}{m_1 + m_2} = \mu r^2 \end{aligned}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

(c) Moment of inertia about the y-axis: By symmetry, rotation about any axis perpendicular to the molecular bond gives the same result:

$$I_y = I_x = \mu r^2$$

Summary of Principal Moments of Inertia: For a rigid diatomic molecule, the three principal moments of inertia are:

$$I_z = 0 \quad \text{(along molecular axis)} \tag{9}$$
$$I_x = I_y = \mu r^2 \quad \text{(perpendicular to molecular axis)} \tag{10}$$

where $\mu = \frac{m_1 m_2}{m_1 + m_2}$ is the reduced mass.

Physical Significance:

- The zero moment about the molecular axis reflects that point masses have no rotational inertia about their connecting line.
- The two equal perpendicular moments arise from the cylindrical symmetry of the linear molecule.
- The reduced mass form μr^2 is fundamental in rotational spectroscopy and appears in the rotational energy levels: $E_J = \frac{\hbar^2 J(J+1)}{2I}$.

(ii) A proton is 1837 times heavier than an electron. Find the center of mass of a hydrogen atom.

Introduction: A hydrogen atom consists of a proton and an electron. The proton is approximately 1837 times more massive than the electron. While the electron in a hydrogen atom doesn't have a well-defined classical position due to quantum mechanics, we can find the center of mass by considering the average position of the electron or by treating this as a classical two-particle system for pedagogical purposes.

Classical Approach: Let the mass of the electron be m_e, and the mass of the proton be $m_p = 1837\,m_e$.

For a classical treatment, consider the proton at the origin ($x = 0$) and the electron at some distance r from the proton ($x = r$). The center of mass position is given by:

$$x_{\text{COM}} = \frac{m_p x_p + m_e x_e}{m_p + m_e}$$

Substituting the values:

$$x_{\text{COM}} = \frac{1837\,m_e \cdot 0 + m_e \cdot r}{1837\,m_e + m_e} = \frac{m_e r}{1838\,m_e} = \frac{r}{1838}$$

Quantum Mechanical Consideration: In quantum mechanics, the electron doesn't have a definite position, but we can consider its expectation value. For the ground state hydrogen atom, the expectation value of the electron's distance from the proton is $\langle r \rangle = \frac{3a_0}{2}$, where $a_0 = 0.529 \times 10^{-10}$ m is the Bohr radius.

Using this quantum mechanical average:

$$x_{\text{COM}} = \frac{\langle r \rangle}{1838} = \frac{3a_0}{2 \times 1838} \approx \frac{1.59a_0}{1838} \approx 8.65 \times 10^{-4} a_0$$

Numerical Result: Taking $a_0 = 0.529 \times 10^{-10}$ m:

$$x_{\text{COM}} \approx 4.58 \times 10^{-14} \text{ m}$$

Physical Interpretation:

- The center of mass lies extremely close to the proton (less than 0.1% of the Bohr radius away).
- This justifies treating the proton as effectively stationary in many hydrogen atom calculations.
- The result is independent of the specific value of r used - the COM is always at $r/1838$ from the proton.
- This analysis forms the basis for reduced mass concepts in atomic physics, where $\mu = \frac{m_p m_e}{m_p + m_e} \approx m_e$.

Conclusion: The center of mass of a hydrogen atom lies at a distance of approximately $\frac{r}{1838}$ from the proton, where r is the characteristic electron-proton separation. This distance is negligible compared to atomic dimensions, confirming that the proton can be treated as the center of mass in most atomic calculations.

49 Write down Euler's dynamical equations of motion (no derivation) for a rigid body about a fixed point under the action of a torque. Show that the kinetic energy of the torque-free motion is constant.

Introduction: Euler's equations describe the rotational dynamics of a rigid body about a fixed point, accounting for external torques. These equations relate the components of angular velocity and angular momentum in the body-fixed frame aligned with principal axes. In the absence of external torque (torque-free motion), we aim to show that the rotational kinetic energy remains constant.

Solution:

Let I_1, I_2, and I_3 be the principal moments of inertia about the body-fixed axes, and let ω_1, ω_2, and ω_3 be the corresponding components of angular velocity. Let N_1, N_2, and N_3 be the components of the external torque in the body-fixed frame.

Euler's equations of motion (no derivation) are:

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= N_1 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= N_2 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= N_3 \end{aligned}$$

For torque-free motion, the external torques are zero:

$$N_1 = N_2 = N_3 = 0$$

Thus, Euler's equations reduce to:

$$\begin{aligned} I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\ I_2\dot{\omega}_2 + (I_1 - I_3)\omega_3\omega_1 &= 0 \\ I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0 \end{aligned}$$

The rotational kinetic energy T of the rigid body is given by:

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

To show that T is constant, compute its time derivative:

$$\frac{dT}{dt} = I_1\omega_1\dot{\omega}_1 + I_2\omega_2\dot{\omega}_2 + I_3\omega_3\dot{\omega}_3$$

Substitute from Euler's torque-free equations:

$$\begin{aligned} \frac{dT}{dt} &= \omega_1[-(I_3 - I_2)\omega_2\omega_3] + \omega_2[-(I_1 - I_3)\omega_3\omega_1] + \omega_3[-(I_2 - I_1)\omega_1\omega_2] \\ &= -(I_3 - I_2)\omega_1\omega_2\omega_3 - (I_1 - I_3)\omega_1\omega_2\omega_3 - (I_2 - I_1)\omega_1\omega_2\omega_3 \\ &= -\omega_1\omega_2\omega_3\left[(I_3 - I_2) + (I_1 - I_3) + (I_2 - I_1)\right] \\ &= -\omega_1\omega_2\omega_3 \cdot 0 = 0 \end{aligned}$$

Conclusion: Euler's equations govern the rotation of a rigid body about a fixed point under torque. In the absence of external torque, the rotational kinetic energy remains constant, indicating conservation of energy in torque-free rigid body motion.

50 Where do you find the applications of gyroscopes? A top of mass 0.200 kg consists of a thin disc of radius 0.12 m. A pin of negligible mass is mounted normal to its plane. The pivot under the disc is 0.03 m long. The top spins with its axis making an angle $\theta = 20^\circ$ with the vertical and a precessional angular speed of 2 rad/s. Calculate its spin angular speed.

Applications of Gyroscopes: Gyroscopes find extensive applications across various fields due to their ability to maintain orientation and detect rotational motion:

- **Navigation Systems**: Aircraft, ships, and submarines use gyroscopic compasses and inertial navigation systems (INS) for accurate directional guidance independent of magnetic fields.
- **Aerospace**: Spacecraft and satellites employ gyroscopes for attitude control and stabilization during orbital maneuvers.
- **Consumer Electronics**: Smartphones, tablets, and gaming controllers use MEMS gyroscopes for screen rotation, motion sensing, and gaming applications.
- **Transportation**: Modern vehicles use gyroscopic sensors in electronic stability control (ESC) systems to prevent skidding and rollover.
- **Robotics**: Autonomous robots and drones rely on gyroscopes for balance and orientation control.
- **Scientific Instruments**: Telescopes use gyrostabilized mounts for precise astronomical observations.
- **Military Applications**: Guided missiles and torpedoes use gyroscopic guidance systems for accurate targeting.

Problem Analysis: Given data:

$$m = 0.200\,\text{kg}, \quad R = 0.12\,\text{m}, \quad l = 0.03\,\text{m}$$
$$\theta = 20^\circ, \quad \Omega = 2\,\text{rad/s} \quad \text{(precession angular speed)}$$

Solution: For a symmetric top undergoing steady precession under gravity, the relationship between precession angular velocity Ω and spin angular velocity ω_s is derived from the torque equation:

The gravitational torque about the pivot point is $\tau = mgl\sin\theta$, which causes the angular momentum vector to precess. For steady precession:

$$\tau = \Omega \times L_s = \Omega L_s \sin\theta$$

where $L_s = I_s\omega_s$ is the spin angular momentum. This gives:

$$mgl\sin\theta = \Omega I_s \omega_s \sin\theta$$

Simplifying:

$$\Omega = \frac{mgl}{I_s \omega_s}$$

Solving for the spin angular velocity:

$$\omega_s = \frac{mgl}{I_s \Omega}$$

The moment of inertia of a thin disc about its central axis is:

$$I_s = \frac{1}{2}mR^2 = \frac{1}{2} \times 0.200 \times (0.12)^2 = 0.00144\,\text{kg}\cdot\text{m}^2$$

Substituting the values:

$$\omega_s = \frac{0.200 \times 9.81 \times 0.03}{0.00144 \times 2} = \frac{0.05886}{0.00288} = 20.44\,\text{rad/s}$$

Verification: We can verify this makes physical sense: the high spin rate (20.44 rad/s) creates sufficient angular momentum to maintain stable precession at the relatively low precession rate (2 rad/s).

Conclusion: The spin angular speed of the top is approximately 20.4 rad/s. This demonstrates the fundamental principle of gyroscopic motion where a rapidly spinning object can maintain stable precession under the influence of gravity, forming the basis for many practical gyroscopic applications listed above.