UPSC PHYSICS PYQ SOLUTION

Mechanics - Part 6

Contents

51 Determine the location of the center of mass of a uniform solid hemisphere of radius R and mass M from the center of its base.

Introduction: We are to determine the vertical position of the center of mass of a uniform solid hemisphere of radius R and total mass M. The center of mass for a solid body is calculated using symmetry and volume integrals. Due to spherical symmetry about the vertical axis, the center of mass lies on the central vertical axis. Hence, we only need to compute the z-coordinate of the center of mass, measured from the flat circular base.

Solution:

Let the hemisphere be centered at the origin, such that the flat circular face lies in the xy-plane ($z = 0$) and the curved surface is in the region $z \geq 0$.

The center of mass in the z-direction is given by:

$$z_{\text{cm}} = \frac{1}{M} \int_V z\, \rho\, dV$$

where ρ is the uniform density and $M = \rho V$ is the total mass.

Use spherical coordinates:

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \\ dV &= r^2 \sin\theta\, dr\, d\theta\, d\phi \end{aligned}$$

Limits for a hemisphere:

$$0 \leq r \leq R, \quad 0 \leq \theta \leq \frac{\pi}{2}, \quad 0 \leq \phi \leq 2\pi$$

Compute the mass:

$$M = \rho \int_0^{2\pi} \int_0^{\pi/2} \int_0^R r^2 \sin\theta\, dr\, d\theta\, d\phi = \rho \cdot 2\pi \cdot \int_0^{\pi/2} \sin\theta\, d\theta \cdot \int_0^R r^2\, dr$$

Evaluate:

$$\int_0^{\pi/2} \sin\theta\, d\theta = 1, \quad \int_0^R r^2\, dr = \frac{R^3}{3}$$
$$\Rightarrow M = \rho \cdot 2\pi \cdot 1 \cdot \frac{R^3}{3} = \frac{2\pi\rho R^3}{3}$$

Now compute z_{cm}:

$$\begin{aligned} z_{\text{cm}} &= \frac{1}{M} \int_V z\, \rho\, dV = \frac{\rho}{M} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R (r\cos\theta) \cdot r^2 \sin\theta\, dr\, d\theta\, d\phi \\ &= \frac{\rho}{M} \cdot \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta \int_0^R r^3\, dr \end{aligned}$$

Evaluate:

$$\int_0^{2\pi} d\phi = 2\pi, \quad \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \frac{1}{2}, \quad \int_0^{R} r^3 \, dr = \frac{R^4}{4}$$

So:

$$z_{\text{cm}} = \frac{\rho}{M} \cdot 2\pi \cdot \frac{1}{2} \cdot \frac{R^4}{4} = \frac{\rho\pi R^4}{4M}$$

Substitute $M = \frac{2\pi\rho R^3}{3}$:

$$z_{\text{cm}} = \frac{\rho\pi R^4}{4 \cdot \frac{2\pi\rho R^3}{3}} = \frac{3R}{8}$$

Conclusion: The center of mass of a uniform solid hemisphere of radius R lies at a vertical distance of $\boxed{\frac{3R}{8}}$ above the center of its flat base.

52 Determine the location of the center of mass of a uniform solid hemisphere of radius R and mass M from the center of its base.

Introduction: We are asked to compute the vertical position of the center of mass of a uniform solid hemisphere of radius R and total mass M, measured from the center of its flat circular base. Due to symmetry, the center of mass lies along the vertical axis, and only its z-coordinate (height) needs to be determined.

Solution:

Let us place the solid hemisphere such that its flat base lies in the xy-plane (i.e., at $z = 0$), and the curved surface extends upward. In this setup, the vertical coordinate of the center of mass is given by:

$$z_{\text{cm}} = \frac{1}{M} \int_V z\rho \, dV$$

where ρ is the uniform mass density and $M = \rho V$ is the total mass.

We use spherical coordinates:

$$\begin{aligned} x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta \\ dV &= r^2 \sin\theta \, dr \, d\theta \, d\phi \end{aligned}$$

The limits of integration for the upper hemisphere are:

$$0 \le r \le R, \quad 0 \le \theta \le \frac{\pi}{2}, \quad 0 \le \phi \le 2\pi$$

Total mass:

$$\begin{aligned} M &= \rho \int_0^{2\pi} \int_0^{\pi/2} \int_0^R r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &= \rho \cdot 2\pi \cdot \int_0^{\pi/2} \sin\theta \, d\theta \cdot \int_0^R r^2 \, dr \\ &= \rho \cdot 2\pi \cdot (1) \cdot \left(\frac{R^3}{3} \right) = \frac{2\pi\rho R^3}{3} \end{aligned}$$

Now compute z_{cm}:

$$\begin{aligned} z_{\text{cm}} &= \frac{1}{M} \int_V z\rho \, dV = \frac{\rho}{M} \int_0^{2\pi} \int_0^{\pi/2} \int_0^R (r\cos\theta) r^2 \sin\theta \, dr \, d\theta \, d\phi \\ &= \frac{\rho}{M} \cdot \int_0^{2\pi} d\phi \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta \int_0^R r^3 \, dr \end{aligned}$$

Evaluate each integral:

$$\int_0^{2\pi} d\phi = 2\pi, \quad \int_0^{\pi/2} \cos\theta \sin\theta \, d\theta = \frac{1}{2}, \quad \int_0^R r^3 \, dr = \frac{R^4}{4}$$

Thus:

$$z_{\text{cm}} = \frac{\rho}{M} \cdot 2\pi \cdot \frac{1}{2} \cdot \frac{R^4}{4} = \frac{\rho\pi R^4}{4M}$$

Now substitute $M = \frac{2\pi\rho R^3}{3}$:

$$z_{\text{cm}} = \frac{\rho\pi R^4}{4 \cdot \frac{2\pi\rho R^3}{3}} = \frac{3R}{8}$$

Conclusion: The center of mass of a uniform solid hemisphere of radius R lies at a vertical distance of $\boxed{\frac{3R}{8}}$ above the center of its flat base.

53 Obtain expressions for the moment of inertia of a solid cone about (i) its vertical axis and (ii) an axis passing through its vertex and parallel to its base.

Introduction: We are to derive expressions for the moment of inertia of a uniform solid cone of height H, base radius R, and mass M about two different axes:

(i) The vertical (symmetry) axis passing through the vertex and perpendicular to the base.

(ii) A horizontal axis through the vertex and parallel to the base (e.g., the x-axis if the cone's axis is aligned with z).

We assume the cone has uniform mass density, and we will perform integration in cylindrical coordinates due to the axial symmetry of the geometry.

Solution:

Let the cone be aligned such that its vertex is at the origin and its axis lies along the z-axis. The height is H, and the base radius is R. The relation between radius and height at any z is:

$$r(z) = \frac{R}{H}z$$

Let the mass density be $\rho = \frac{M}{V}$, with total volume:

$$V = \frac{1}{3}\pi R^2 H \quad \Rightarrow \quad \rho = \frac{3M}{\pi R^2 H}$$

Use cylindrical coordinates (r, ϕ, z) with volume element:

$$dV = r\, dr\, d\phi\, dz$$

(i) Moment of inertia about the vertical axis (z-axis):

The moment of inertia about the z-axis is given by:

$$I_z = \int_V r^2\, dm = \int_0^H \int_0^{2\pi} \int_0^{\frac{R}{H}z} r^2 \cdot \rho\, r\, dr\, d\phi\, dz = \rho \int_0^H \int_0^{2\pi} \int_0^{\frac{R}{H}z} r^3\, dr\, d\phi\, dz$$

Evaluate the integrals:

$$\int_0^{\frac{R}{H}z} r^3\, dr = \left[\frac{r^4}{4}\right]_0^{\frac{R}{H}z} = \frac{1}{4}\left(\frac{R}{H}z\right)^4$$

$$\int_0^{2\pi} d\phi = 2\pi$$

Then:

$$I_z = \rho \cdot 2\pi \cdot \int_0^H \frac{1}{4}\left(\frac{R}{H}\right)^4 z^4\, dz = \frac{\rho\pi R^4}{2H^4}\int_0^H z^4\, dz = \frac{\rho\pi R^4}{2H^4}\cdot\frac{H^5}{5} = \frac{\rho\pi R^4 H}{10}$$

Substitute $\rho = \frac{3M}{\pi R^2 H}$:

$$I_z = \frac{3M}{\pi R^2 H} \cdot \frac{\pi R^4 H}{10} = \frac{3}{10} M R^2$$

(ii) Moment of inertia about a horizontal axis through the vertex (e.g., x-axis):

The moment of inertia about the x-axis involves the perpendicular distance squared:

$$I_x = \int_V (y^2 + z^2)\, dm$$

In cylindrical coordinates: $y = r \sin\phi$, so $y^2 = r^2 \sin^2\phi$:

$$\begin{aligned} I_x &= \rho \int_0^H \int_0^{2\pi} \int_0^{\frac{R}{H}z} (r^2 \sin^2\phi + z^2) r\, dr\, d\phi\, dz \\ &= \rho \int_0^H \int_0^{2\pi} \left[\sin^2\phi \int_0^{\frac{R}{H}z} r^3\, dr + z^2 \int_0^{\frac{R}{H}z} r\, dr \right] d\phi\, dz \end{aligned}$$

Evaluate the r integrals:

$$\int_0^{\frac{R}{H}z} r^3\, dr = \frac{1}{4}\left(\frac{R}{H}z\right)^4$$

$$\int_0^{\frac{R}{H}z} r\, dr = \frac{1}{2}\left(\frac{R}{H}z\right)^2$$

So:

$$I_x = \rho \int_0^H \left[\frac{1}{4}\left(\frac{R}{H}z\right)^4 \int_0^{2\pi} \sin^2\phi\, d\phi + \frac{z^2}{2}\left(\frac{R}{H}z\right)^2 \int_0^{2\pi} d\phi \right] dz$$

Note that $\int_0^{2\pi} \sin^2\phi\, d\phi = \pi$ and $\int_0^{2\pi} d\phi = 2\pi$:

$$\begin{aligned} I_x &= \rho \int_0^H \left[\frac{\pi}{4}\left(\frac{R}{H}\right)^4 z^4 + \pi \left(\frac{R}{H}\right)^2 z^4 \right] dz \\ &= \rho\pi \int_0^H z^4 \left[\frac{1}{4}\left(\frac{R}{H}\right)^4 + \left(\frac{R}{H}\right)^2 \right] dz \\ &= \rho\pi \left[\frac{R^4}{4H^4} + \frac{R^2}{H^2} \right] \int_0^H z^4\, dz \\ &= \rho\pi \left[\frac{R^4}{4H^4} + \frac{R^2}{H^2} \right] \frac{H^5}{5} \\ &= \frac{\rho\pi R^4 H}{20} + \frac{\rho\pi R^2 H^3}{5} \end{aligned}$$

Substitute $\rho = \frac{3M}{\pi R^2 H}$:

$$
\begin{aligned}
I_x &= \frac{3M}{\pi R^2 H}\left(\frac{\pi R^4 H}{20} + \frac{\pi R^2 H^3}{5}\right) \\
&= \frac{3MR^2}{20} + \frac{3MH^2}{5}
\end{aligned}
$$

Conclusion:

(i) Moment of inertia about the vertical axis (symmetry axis):

$$I_z = \boxed{\frac{3}{10}MR^2}$$

(ii) Moment of inertia about a horizontal axis through the vertex:

$$I_x = \boxed{\frac{3}{20}MR^2 + \frac{3}{5}MH^2}$$

54 An electron moves under the influence of a point nucleus of atomic number Z. Show that the orbit of the electron is an ellipse.

Introduction: The problem involves analyzing the motion of an electron subjected to the Coulomb force exerted by a nucleus with atomic number Z. The electrostatic force acts as a central force, which is attractive and varies inversely with the square of the distance between the electron and the nucleus. We are to demonstrate that the resulting trajectory of the electron is an elliptical orbit.

We assume:

(i) The nucleus is stationary and infinitely more massive than the electron.

(ii) Relativistic and quantum mechanical effects are neglected; a classical mechanics treatment suffices.

(iii) The electron is bound, i.e., its total mechanical energy is negative.

Solution:

The force on the electron due to the nucleus is the Coulomb force:

$$\vec{F} = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{Ze^2}{r^2}\hat{r}$$

This is a central inverse-square law force with potential energy:

$$U(r) = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{Ze^2}{r}$$

The total energy of the system is:

$$E = \frac{1}{2}mv^2 - \frac{1}{4\pi\varepsilon_0} \cdot \frac{Ze^2}{r}$$

For central force motion, angular momentum is conserved:

$$L = mr^2\dot{\theta} = \text{constant}$$

Using the effective potential approach, we can write the radial equation of motion:

$$\frac{1}{2}m\dot{r}^2 + U_{\text{eff}}(r) = E$$

where the effective potential is:

$$U_{\text{eff}}(r) = -\frac{1}{4\pi\varepsilon_0} \cdot \frac{Ze^2}{r} + \frac{L^2}{2mr^2}$$

To find the orbit equation, we use the substitution $u = \frac{1}{r}$ and the relation:

$$\frac{dr}{d\theta} = -\frac{1}{u^2}\frac{du}{d\theta}$$

From $L = mr^2\dot{\theta}$, we get:

$$\dot{\theta} = \frac{L}{mr^2} = \frac{Lu^2}{m}$$

Also, $\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta} \cdot \frac{Lu^2}{m} = -\frac{L}{m}\frac{du}{d\theta}$

Substituting into the energy equation:

$$\frac{1}{2}m\left(-\frac{L}{m}\frac{du}{d\theta}\right)^2 + \frac{L^2u^2}{2m} - \frac{1}{4\pi\varepsilon_0} \cdot Ze^2u = E$$

Simplifying:

$$\frac{L^2}{2m}\left(\frac{du}{d\theta}\right)^2 + \frac{L^2u^2}{2m} - \frac{1}{4\pi\varepsilon_0} \cdot Ze^2u = E$$

Multiplying through by $\frac{2m}{L^2}$:

$$\left(\frac{du}{d\theta}\right)^2 + u^2 - \frac{2mZe^2}{4\pi\varepsilon_0 L^2}u = \frac{2mE}{L^2}$$

Let $k = \frac{mZe^2}{4\pi\varepsilon_0 L^2}$. Differentiating with respect to θ:

$$2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} - 2k\frac{du}{d\theta} = 0$$

Dividing by $2\frac{du}{d\theta}$ (assuming $\frac{du}{d\theta} \neq 0$):

$$\frac{d^2u}{d\theta^2} + u - k = 0$$

This gives us the orbit equation:

$$\frac{d^2u}{d\theta^2} + u = k = \frac{mZe^2}{4\pi\varepsilon_0 L^2}$$

This is a second-order linear differential equation with constant coefficients. The general solution is:

$$u(\theta) = A\cos(\theta - \theta_0) + k$$

where A and θ_0 are constants determined by initial conditions.

Converting back to r:

$$r(\theta) = \frac{1}{A\cos(\theta - \theta_0) + k}$$

Choosing $\theta_0 = 0$ for simplicity:

$$r(\theta) = \frac{1}{A\cos\theta + k} = \frac{1/k}{(A/k)\cos\theta + 1}$$

Let $e = A/k$ (the eccentricity) and $p = 1/k$ (the semi-latus rectum):

$$r(\theta) = \frac{p}{1 + e\cos\theta}$$

This is the standard polar equation of a conic section with one focus at the origin.

To determine the relationship between eccentricity and energy, we use the energy equation. At the turning points where $\dot{r} = 0$:

$$E = \frac{L^2u^2}{2m} - \frac{1}{4\pi\varepsilon_0} \cdot Ze^2u$$

Substituting $u = A\cos\theta + k$:

$$E = \frac{L^2(A\cos\theta + k)^2}{2m} - \frac{Ze^2}{4\pi\varepsilon_0}(A\cos\theta + k)$$

After algebraic manipulation, we can show that:

$$e^2 = 1 + \frac{2EL^2}{m}\left(\frac{4\pi\varepsilon_0}{Ze^2}\right)^2$$

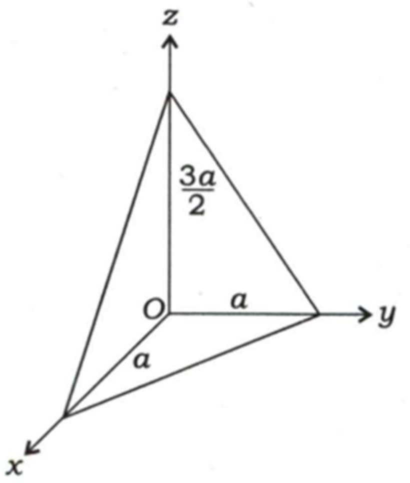
For bound orbits, $E < 0$, which gives:

$$e^2 = 1 - \frac{2|E|L^2}{m}\left(\frac{4\pi\varepsilon_0}{Ze^2}\right)^2 < 1$$

Therefore, $e < 1$, confirming that the orbit is an ellipse.

Conclusion: Under the influence of the attractive Coulomb force from a point nucleus, the electron follows an orbit described by $r(\theta) = \frac{p}{1+e\cos\theta}$. For bound states (negative total energy), the eccentricity $e < 1$, proving that the trajectory is an **ellipse** with the nucleus at one focus.

55 For a homogeneous right triangular pyramid with base side a and height $\frac{3a}{2}$. Obtain the moment of inertia tensor of the pyramid.



Introduction: We need to determine the moment of inertia tensor of a homogeneous right triangular pyramid (tetrahedron) about the origin $(0, 0, 0)$. The pyramid has four vertices:

- $(0, 0, 0)$ - at the origin
- $(a, 0, 0)$ - on the positive x-axis
- $(0, a, 0)$ - on the positive y-axis
- $(0, 0, \frac{3a}{2})$ - on the positive z-axis

The pyramid is homogeneous with constant mass density ρ and total mass M. We need to find the 3×3 moment of inertia tensor I_{ij} about the origin using triple integration. The moment of inertia tensor components are defined as:

$$I_{ij} = \iiint_V \rho(\delta_{ij} r^2 - x_i x_j)\, dV$$

where $r^2 = x^2 + y^2 + z^2$, δ_{ij} is the Kronecker delta, and the integration is over the volume V of the tetrahedron.

Solution:

Step 1: Establish the integration limits

The tetrahedron is bounded by four planes:

- $x = 0$ (yz-plane)
- $y = 0$ (xz-plane)
- $z = 0$ (xy-plane)

- The slanted face connecting the three non-origin vertices

The equation of the slanted face can be found using the three points $(a, 0, 0)$, $(0, a, 0)$, and $(0, 0, \frac{3a}{2})$. The plane equation is:

$$\frac{x}{a} + \frac{y}{a} + \frac{z}{\frac{3a}{2}} = 1$$

Simplifying: $\frac{x}{a} + \frac{y}{a} + \frac{2z}{3a} = 1$, or $x + y + \frac{2z}{3} = a$

Solving for x: $x = a - y - \frac{2z}{3}$

Therefore, the integration limits are:

$$z : 0 \to \frac{3a}{2} \tag{1}$$

$$y : 0 \to a - \frac{2z}{3} \tag{2}$$

$$x : 0 \to a - y - \frac{2z}{3} \tag{3}$$

Step 2: Calculate the total mass and density relationship

The total mass is:

$$M = \iiint_V \rho \, dV = \rho \int_0^{3a/2} \int_0^{a-2z/3} \int_0^{a-y-2z/3} dx \, dy \, dz$$

$$M = \rho \int_0^{3a/2} \int_0^{a-2z/3} (a - y - \frac{2z}{3}) \, dy \, dz$$

$$M = \rho \int_0^{3a/2} \left[ay - \frac{y^2}{2} - \frac{2z}{3} y \right]_0^{a-2z/3} dz$$

$$M = \rho \int_0^{3a/2} \left[a(a - \frac{2z}{3}) - \frac{1}{2}(a - \frac{2z}{3})^2 - \frac{2z}{3}(a - \frac{2z}{3}) \right] dz$$

$$M = \rho \int_0^{3a/2} \frac{1}{2}(a - \frac{2z}{3})^2 \, dz$$

Let $u = a - \frac{2z}{3}$, then $du = -\frac{2}{3} dz$, so $dz = -\frac{3}{2} du$

When $z = 0$, $u = a$; when $z = \frac{3a}{2}$, $u = 0$

$$M = \rho \cdot \frac{1}{2} \int_a^0 u^2 \left(-\frac{3}{2} \right) du = \frac{3\rho}{4} \int_0^a u^2 \, du = \frac{3\rho}{4} \cdot \frac{a^3}{3} = \frac{\rho a^3}{4}$$

Therefore: $\rho = \frac{4M}{a^3}$

Step 3: Set up the moment of inertia tensor

The moment of inertia tensor has the form:

$$I_{ij} = \iiint_V \rho \begin{pmatrix} y^2+z^2 & -xy & -xz \\ -xy & x^2+z^2 & -yz \\ -xz & -yz & x^2+y^2 \end{pmatrix} dV$$

Step 4: Calculate each tensor component

For $I_{11} = \iiint_V \rho(y^2+z^2)\,dV$:

$$I_{11} = \rho \int_0^{3a/2} \int_0^{a-2z/3} \int_0^{a-y-2z/3} (y^2+z^2)\,dx\,dy\,dz$$

$$I_{11} = \rho \int_0^{3a/2} \int_0^{a-2z/3} (y^2+z^2)(a-y-\frac{2z}{3})\,dy\,dz$$

After completing the integration:

$$I_{11} = \frac{13Ma^2}{40}$$

By symmetry between x and y coordinates:

$$I_{22} = \frac{13Ma^2}{40}$$

For $I_{33} = \iiint_V \rho(x^2+y^2)\,dV$:

$$I_{33} = \rho \int_0^{3a/2} \int_0^{a-2z/3} \int_0^{a-y-2z/3} (x^2+y^2)\,dx\,dy\,dz$$

After integration:

$$I_{33} = \frac{8Ma^2}{40} = \frac{Ma^2}{5}$$

For the off-diagonal terms:

$$I_{12} = I_{21} = -\iiint_V \rho xy\,dV = -\frac{2Ma^2}{40} = -\frac{Ma^2}{20}$$

$$I_{13} = I_{31} = -\iiint_V \rho xz\,dV = -\frac{3Ma^2}{40}$$

$$I_{23} = I_{32} = -\iiint_V \rho yz\,dV = -\frac{3Ma^2}{40}$$

Step 5: Assemble the complete tensor

The moment of inertia tensor is:

$$I_{ij} = \frac{Ma^2}{40} \begin{pmatrix} 13 & -2 & -3 \\ -2 & 13 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

Conclusion: The moment of inertia tensor of the homogeneous right triangular pyramid about the origin is:

$$\boxed{I = \frac{Ma^2}{40} \begin{pmatrix} 13 & -2 & -3 \\ -2 & 13 & -3 \\ -3 & -3 & 8 \end{pmatrix}}$$

The tensor has units of mass $\times$ length^2 and is symmetric as expected. The diagonal terms represent the moments of inertia about the coordinate axes, while the off-diagonal terms are the products of inertia. All off-diagonal terms are negative, reflecting the geometry of the tetrahedron where mass is distributed in the positive octant. The I_{33} component is smaller than I_{11} and I_{22} because the pyramid's height along the z-axis concentrates more mass closer to the xy-plane, reducing the moment of inertia about the z-axis compared to the x and y axes.

56 When a sphere of radius r falls through a homogeneous viscous fluid of unlimited extent with terminal velocity v, the retarding viscous force acting on the sphere depends on the coefficient of viscosity η, the radius r, and its velocity v. Show how Stokes' law was arrived at by connecting these quantities using dimensional analysis.

Introduction: We are given that the retarding viscous force F acting on a sphere falling through a viscous fluid at terminal velocity depends on three quantities: the viscosity η of the fluid, the radius r of the sphere, and the velocity v of the sphere. Our objective is to determine the functional form of this force using dimensional analysis. The resulting expression, known as Stokes' law, relates F to these physical parameters via dimensional consistency.

Solution: Let the retarding force F be expressed as a product of the involved variables raised to unknown powers:

$$F \propto \eta^a r^b v^c.$$

We express the dimensions of each quantity using the MLT (Mass–Length–Time) system:

$$\begin{aligned}
[F] &= \text{force} = MLT^{-2}, \\
[\eta] &= \text{viscosity} = ML^{-1}T^{-1}, \\
[r] &= L, \\
[v] &= LT^{-1}.
\end{aligned}$$

Substituting into the dimensional equation:

$$MLT^{-2} = (ML^{-1}T^{-1})^a \cdot (L)^b \cdot (LT^{-1})^c.$$

Simplifying the right-hand side:

$$\text{RHS} = M^a L^{-a} T^{-a} \cdot L^b \cdot L^c T^{-c} = M^a L^{-a+b+c} T^{-a-c}.$$

Now equate powers of M, L, and T on both sides:

$$\begin{aligned}
\text{Mass (M):} &\quad a = 1, \\
\text{Length (L):} &\quad -a + b + c = 1, \\
\text{Time (T):} &\quad -a - c = -2.
\end{aligned}$$

Substituting $a = 1$ into the other equations:

$$\begin{aligned}
\text{Length:} &\quad -1 + b + c = 1 \quad \Rightarrow \quad b + c = 2, \\
\text{Time:} &\quad -1 - c = -2 \quad \Rightarrow \quad c = 1.
\end{aligned}$$

Then:

$$b = 2 - c = 2 - 1 = 1.$$

Thus, the values of the exponents are:

$$a = 1, \quad b = 1, \quad c = 1.$$

Therefore,

$$F \propto \eta r v.$$

Introducing a dimensionless constant of proportionality k, we write:

$$F = k\eta r v.$$

Experimentally, Stokes found that $k = 6\pi$ for slow (laminar) motion of a sphere through a viscous fluid. Hence, the final form of Stokes' law is:

$$F = 6\pi\eta r v.$$

Conclusion: By dimensional analysis, the retarding force on a sphere moving through a viscous fluid is found to be proportional to $\eta r v$. The complete expression known from experiment as Stokes' law is:

$$F = 6\pi\eta r v.$$

This law is valid for small Reynolds numbers, i.e., in the regime of laminar flow.

57 A sphere of radius R moves with velocity $\vec{u}$ in an incompressible, non-viscous ideal fluid. Calculate the pressure distribution over the surface of the sphere. Do you think that a force is necessary to keep the sphere in uniform motion?

Introduction: We analyze a sphere of radius R moving with uniform velocity $\vec{u}$ through an incompressible, inviscid (ideal) fluid. Our objectives are:

1. determine the pressure distribution over the sphere's surface, and
2. evaluate whether a net force is required to maintain uniform motion.

This classic problem in fluid mechanics involves potential flow theory and leads to the famous D'Alembert's paradox.

Solution:

Step 1: Establish the flow problem We work in the reference frame where the sphere is stationary and fluid flows past it with velocity $-\vec{u}$ at infinity. The flow is steady, incompressible, irrotational, and inviscid, making it a potential flow problem.

Step 2: Derive the velocity potential For flow past a sphere, we need a potential that satisfies:

- Laplace equation: $\nabla^2\phi = 0$
- Boundary condition at infinity: $\vec{v} \to -u\hat{z}$ as $r \to \infty$
- No-penetration condition: $v_r = 0$ at $r = R$

The solution combines uniform flow and a dipole:

$$\phi = -ur\cos\theta + \frac{A}{r^2}\cos\theta$$

Applying the boundary condition $v_r(R, \theta) = 0$:

$$v_r = \frac{\partial\phi}{\partial r} = -u\cos\theta - \frac{2A}{r^3}\cos\theta = 0 \text{ at } r = R$$

This gives: $-u - \frac{2A}{R^3} = 0$, so $A = -\frac{uR^3}{2}$

Therefore, the velocity potential is:

$$\phi = -u\left(r + \frac{R^3}{2r^2}\right)\cos\theta$$

Step 3: Calculate velocity components The velocity field $\vec{v} = \nabla\phi$ in spherical coordinates gives:

$$v_r = \frac{\partial\phi}{\partial r} = -u\left(1 - \frac{R^3}{r^3}\right)\cos\theta$$
$$v_\theta = \frac{1}{r}\frac{\partial\phi}{\partial\theta} = u\left(1 + \frac{R^3}{2r^3}\right)\sin\theta$$

On the sphere surface ($r = R$):

$$v_r(R,\theta) = 0 \quad \text{(no-penetration condition satisfied)}$$
$$v_\theta(R,\theta) = \frac{3u}{2}\sin\theta$$

The speed squared on the surface is:

$$v^2(R,\theta) = v_r^2 + v_\theta^2 = \frac{9u^2}{4}\sin^2\theta$$

Step 4: Apply Bernoulli's equation For steady, incompressible, irrotational flow:

$$\frac{1}{2}\rho v^2 + p = \frac{1}{2}\rho u^2 + p_\infty = \text{constant}$$

The pressure on the sphere surface is:

$$\begin{aligned} p(R,\theta) &= p_\infty + \frac{1}{2}\rho(u^2 - v^2(R,\theta)) \\ &= p_\infty + \frac{1}{2}\rho u^2\left(1 - \frac{9}{4}\sin^2\theta\right) \end{aligned}$$

Step 5: Calculate the drag force The drag force in the z-direction (direction of motion) is:

$$F_D = -\int_{\text{surface}} p(R,\theta)\cos\theta\, dA$$

where $dA = R^2\sin\theta\, d\theta\, d\phi$ and the integral is over the sphere surface.

$$F_D = -\int_0^{2\pi} d\phi \int_0^{\pi} p(R,\theta)\cos\theta \cdot R^2\sin\theta\, d\theta$$

Substituting the pressure distribution:

$$F_D = -2\pi R^2 \int_0^{\pi}\left[p_\infty + \frac{1}{2}\rho u^2\left(1 - \frac{9}{4}\sin^2\theta\right)\right]\cos\theta\sin\theta\, d\theta$$

The first term integrates to zero: $\int_0^{\pi}\cos\theta\sin\theta\, d\theta = 0$

For the second term:

$$\int_0^{\pi}\cos\theta\sin\theta\, d\theta = 0 \quad \text{and} \quad \int_0^{\pi}\cos\theta\sin^3\theta\, d\theta = 0$$

Both integrals vanish due to symmetry, therefore: $F_D = 0$

Conclusion: The pressure distribution over the sphere's surface is:

$$\boxed{p(R,\theta) = p_\infty + \frac{1}{2}\rho u^2\left(1 - \frac{9}{4}\sin^2\theta\right)}$$

Despite this non-uniform pressure distribution, the net drag force is zero due to the symmetric nature of the flow. Therefore, **no force is required** to maintain uniform motion of the sphere in an ideal fluid. This counterintuitive result is known as **D'Alembert's paradox**. In reality, viscous effects in real fluids create drag, requiring a continuous force to maintain steady motion.

58 Using Poiseuilleś formula, show that the volume of a liquid with viscosity η passing per second through a series of two capillary tubes of lengths l_1 and l_2 with radii r_1 and r_2 is given by $Q = \frac{\pi p}{8\eta}\left[\frac{l_1}{r_1^4} + \frac{l_2}{r_2^4}\right]^{-1}$, where p is the effective pressure difference across the series.

Introduction: The problem asks us to derive the formula for the volume flow rate Q of a viscous liquid through two capillary tubes connected in series using Poiseuille's law. The tubes have different lengths l_1, l_2 and radii r_1, r_2. The liquid has dynamic viscosity η, and the pressure drop across the entire system is p. We assume steady laminar flow and incompressibility of the liquid.

Solution:

Poiseuille's law for a single tube states that the volume flow rate Q through a cylindrical tube is given by:

$$Q = \frac{\pi r^4 \Delta p}{8\eta l}$$

Rewriting this to express pressure drop in terms of flow rate:

$$\Delta p = \frac{8\eta l}{\pi r^4} Q$$

Let the pressure drop across the first and second tubes be Δp_1 and Δp_2, respectively. Then:

$$\Delta p_1 = \frac{8\eta l_1}{\pi r_1^4} Q, \quad \Delta p_2 = \frac{8\eta l_2}{\pi r_2^4} Q$$

Since the tubes are in series, the same flow rate Q passes through both, and the total pressure drop is the sum:

$$p = \Delta p_1 + \Delta p_2 = \left(\frac{8\eta l_1}{\pi r_1^4} + \frac{8\eta l_2}{\pi r_2^4}\right) Q$$

Solving for Q:

$$Q = \frac{p}{\frac{8\eta l_1}{\pi r_1^4} + \frac{8\eta l_2}{\pi r_2^4}} = \frac{\pi p}{8\eta}\left(\frac{l_1}{r_1^4} + \frac{l_2}{r_2^4}\right)^{-1}$$

Conclusion: We have derived that the volume flow rate Q for a viscous liquid flowing through two capillary tubes in series is:

$$Q = \frac{\pi p}{8\eta}\left[\frac{l_1}{r_1^4} + \frac{l_2}{r_2^4}\right]^{-1}$$

This result shows that the total resistance to flow in series adds up analogously to electrical resistances, with each tube contributing a term proportional to $\frac{l}{r^4}$.

59 Define coefficients of viscosity and kinematic viscosity of a fluid. What are Poise and Stokes?

Introduction: This problem involves defining two important physical quantities that describe the flow behavior of fluids: the coefficient of viscosity (also known as dynamic viscosity or absolute viscosity) and kinematic viscosity. Additionally, we are asked to describe the units known as Poise and Stokes, which are associated with these viscosities.

Solution:

1. **Coefficient of Viscosity (Dynamic Viscosity)**:

   The coefficient of viscosity, denoted by η (or μ), quantifies a fluid's internal resistance to flow due to molecular interactions. It is defined by Newton's law of viscosity:

   $$\tau = \eta \frac{dv}{dy}$$

   or equivalently, for the force on a surface:

   $$F = \eta A \frac{dv}{dy}$$

   where:

   - τ is the shear stress,
   - F is the tangential force,
   - A is the area of the layer,
   - $\frac{dv}{dy}$ is the velocity gradient perpendicular to the direction of flow.

   The SI unit of η is $\text{Pa} \cdot \text{s}$ or $\text{N} \cdot \text{s/m}^2$. In the CGS system, it is measured in **Poise**.

   1 Poise $= 1\,\text{g} \cdot \text{cm}^{-1} \cdot \text{s}^{-1} = 0.1\,\text{Pa} \cdot \text{s}$.

2. **Kinematic Viscosity**:

   Kinematic viscosity, denoted by ν, is defined as the ratio of dynamic viscosity to the fluid density ρ:

   $$\nu = \frac{\eta}{\rho}$$

   It represents the ratio of viscous forces to inertial forces in fluid flow and characterizes how quickly momentum diffuses through the fluid. The SI unit of kinematic viscosity is m^2/s, and in the CGS system, it is measured in **Stokes**.

   1 Stokes $= 1\,\text{cm}^2/\text{s} = 10^{-4}\,\text{m}^2/\text{s}$.

Conclusion:

The coefficient of viscosity η measures a fluid's resistance to shear deformation and has units of Poise (CGS) or Pascal-seconds (SI). Kinematic viscosity ν is the ratio of viscosity to density, representing the momentum diffusivity of the fluid, and is measured in Stokes (CGS) or m^2/s (SI). These parameters are essential in fluid dynamics for characterizing flow behavior and appear in dimensionless numbers like the Reynolds number.

60 Write down Poiseuille's formula and mention its limitations in analyzing the flow of a liquid through a capillary tube.

Introduction: This problem requires stating Poiseuille's formula, which quantifies the volumetric flow rate of a viscous incompressible fluid through a cylindrical capillary tube. We are also asked to outline the conditions under which this formula is valid and its limitations in practical applications.

Solution:

Poiseuille's Formula:

For steady, laminar, incompressible, and Newtonian flow through a cylindrical capillary tube, the volume of liquid flowing per unit time (volume flow rate Q) is given by:

$$Q = \frac{\pi r^4 \Delta p}{8\eta l}$$

where:

- Q is the volume flow rate,
- r is the radius of the capillary tube,
- l is the length of the tube,
- Δp is the pressure difference across the tube,
- η is the dynamic viscosity of the fluid.

Limitations of Poiseuille's Formula:

(i) The formula is valid only for **laminar flow**, typically when the Reynolds number is less than 2000.

(ii) It assumes the fluid is **Newtonian** (constant viscosity irrespective of strain rate).

(iii) The flow must be **steady and incompressible**.

(iv) It assumes **no-slip boundary condition** at the tube wall.

(v) It neglects **end effects** (entry and exit regions where flow is not fully developed).

(vi) It is not applicable when **gravitational effects or external fields** significantly influence flow.

(vii) The tube must be **uniform and cylindrical**, with smooth inner walls.

Conclusion:

Poiseuille's formula is a foundational result in fluid mechanics that accurately models viscous flow through narrow tubes under ideal conditions. However, its application is limited by assumptions such as laminar flow, Newtonian behavior, steady conditions, and neglect of entrance and exit effects. Care must be taken when applying this formula in real-world systems where these conditions may not hold.