UPSC PHYSICS PYQ SOLUTION

Quantum Mechanics - Part 1

Contents

1 Derive Bohr's angular momentum quantization condition in Bohr's atomic model from the concept of de Broglie waves. (2010)

Introduction: Bohr's atomic model introduces the concept of quantized angular momentum for electrons orbiting the nucleus. This concept is fundamentally linked to the wave nature of electrons as described by de Broglie.

Solution: According to de Broglie, the wavelength of an electron is given by:

$$\lambda = \frac{h}{p}$$

where h is Planck's constant and p is the momentum of the electron.

For an electron in a circular orbit of radius r, the circumference must be an integral multiple of the de Broglie wavelength:

$$2\pi r = n\lambda$$

Substituting the de Broglie wavelength:

$$2\pi r = n\frac{h}{p}$$

Since $p = mv$ for an electron of mass m and velocity v:

$$2\pi r = n\frac{h}{mv}$$

Rearranging for the angular momentum L:

$$L = mvr = n\frac{h}{2\pi}$$

Thus, the angular momentum is quantized:

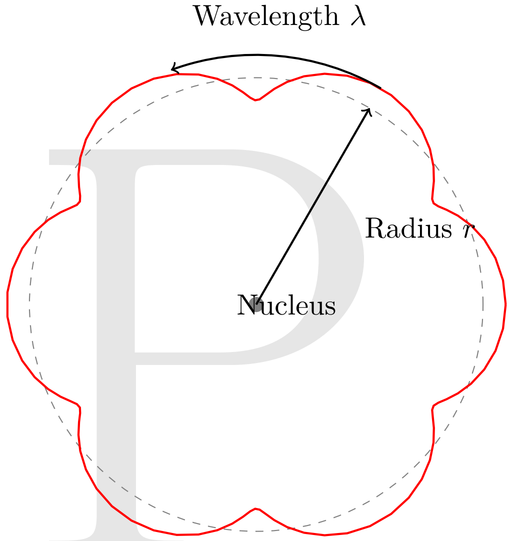

Figure 1: Quantization of angular momentum

$$L = n\hbar$$

where $\hbar = \frac{h}{2\pi}$.

Conclusion: Bohr's quantization of angular momentum provides a fundamental insight into the discrete nature of atomic energy levels, leading to the explanation of atomic spectra.

2 Calculate the wavelength of de Broglie waves associated with electrons accelerated through a potential difference of 200 Volts. (2011)

Introduction: The concept of de Broglie wavelength states that a particle also behaves as a wave, whose wavelength can be calculated when they are accelerated by a known potential difference.

Solution: The kinetic energy gained by the electron is:

$$eV = \frac{1}{2}mv^2$$

where e is the electron charge, V is the potential difference, m is the electron mass, and v is the velocity.

Rearranging for v:

$$v = \sqrt{\frac{2eV}{m}}$$

The de Broglie wavelength is given by:

$$\lambda = \frac{h}{mv}$$

Substituting v:

$$\lambda = \frac{h}{m\sqrt{\frac{2eV}{m}}} = \frac{h}{\sqrt{2meV}}$$

Using $h = 6.626 \times 10^{-34}$ Js, $m = 9.109 \times 10^{-31}$ kg, and $e = 1.602 \times 10^{-19}$ C:

$$\lambda = \frac{6.626 \times 10^{-34}}{\sqrt{2 \times 9.109 \times 10^{-31} \times 1.602 \times 10^{-19} \times 200}}$$

$$\lambda \approx 8.6 \times 10^{-12} \text{ meters}$$

Conclusion: The wavelength of de Broglie waves associated with electrons accelerated through 200 Volts is approximately 8.6×10^{-12} meters, which highlights the wave nature of electrons at the atomic scale.

3 Estimate the size of the hydrogen atom and the ground state energy from the uncertainty principle

Introduction: We aim to estimate the characteristic size of the hydrogen atom and its ground state energy using Heisenberg's uncertainty principle. We will model the electron bound to the proton via Coulomb attraction, and apply quantum mechanical uncertainty relations to find approximate expressions for both the radius and the minimum energy of the electron in its ground state.

Given:

We estimate the electrons position uncertainty as the size of the atom r, and relate the momentum uncertainty Δp via the uncertainty relation $\Delta x\,\Delta p \sim \hbar$.

Solution:

From the uncertainty principle:

$$\Delta x\,\Delta p \sim \hbar \Rightarrow \Delta p \sim \frac{\hbar}{r}$$

Assuming $\Delta p \sim p$, we estimate the kinetic energy as:

$$K \sim \frac{p^2}{2m_e} \sim \frac{\hbar^2}{2m_e r^2}$$

The potential energy due to the Coulomb attraction between electron and proton is:

$$U \sim -\frac{k_e e^2}{r}$$

Hence, the total energy is approximately:

$$E(r) \sim \frac{\hbar^2}{2m_e r^2} - \frac{k_e e^2}{r}$$

To find the equilibrium (minimum energy), we differentiate $E(r)$ with respect to r and set to zero:

$$\frac{dE}{dr} = -\frac{\hbar^2}{m_e r^3} + \frac{k_e e^2}{r^2} = 0$$

Solving:

$$\frac{\hbar^2}{m_e r^3} = \frac{k_e e^2}{r^2} \Rightarrow r = \frac{\hbar^2}{m_e k_e e^2}$$

Substituting values:

$$r = \frac{(1.05 \times 10^{-34})^2}{(9.11 \times 10^{-31})(8.99 \times 10^{9})(1.60 \times 10^{-19})^2}$$

Calculating:

$$\begin{aligned} r &\approx \frac{1.10 \times 10^{-68}}{(9.11 \times 10^{-31})(8.99 \times 10^{9})(2.56 \times 10^{-38})} \\ &\approx \frac{1.10 \times 10^{-68}}{2.09 \times 10^{-58}} \\ &\approx 5.26 \times 10^{-11}\,\mathrm{m} \end{aligned}$$

This is approximately the Bohr radius.

Now substitute r back into $E(r)$ to get the ground state energy:

$$E \approx \frac{\hbar^2}{2m_e r^2} - \frac{k_e e^2}{r}$$

Compute each term:

$$\begin{aligned}
\frac{\hbar^2}{2m_e r^2} &\approx \frac{1.10 \times 10^{-68}}{2 \cdot 9.11 \times 10^{-31} \cdot (5.26 \times 10^{-11})^2} \approx 2.18 \times 10^{-18}\,\mathrm{J} \\
\frac{k_e e^2}{r} &\approx \frac{8.99 \times 10^{9} \cdot (1.60 \times 10^{-19})^2}{5.26 \times 10^{-11}} \approx 4.36 \times 10^{-18}\,\mathrm{J}
\end{aligned}$$

Thus,

$$E \approx 2.18 \times 10^{-18} - 4.36 \times 10^{-18} = -2.18 \times 10^{-18}\,\mathrm{J}$$

Convert to electronvolts:

$$E \approx \frac{-2.18 \times 10^{-18}\,\mathrm{J}}{1.60 \times 10^{-19}\,\mathrm{J/eV}} \approx -13.6\,\mathrm{eV}$$

Conclusion:

Using the uncertainty principle, we estimate:

- The size (radius) of the hydrogen atom: $r \approx 5.26 \times 10^{-11}\,\mathrm{m}$ (Bohr radius)
- The ground state energy: $E_0 \approx -13.6\,\mathrm{eV}$

These estimates agree remarkably well with the results from the full quantum mechanical treatment of the hydrogen atom, illustrating the power of the uncertainty principle in deriving fundamental atomic properties.

4 Use the uncertainty principle to estimate the ground state energy of a linear harmonic oscillator. (2012)

Introduction: The uncertainty principle, formulated by Werner Heisenberg, states that two conjugate pair (which do not commute) in quantum mechanics can never be precisely measured simultaneously. In this case it states that energy and time cant be exactly determined simultaneously.

Solution: For a harmonic oscillator, the potential energy is given by:

$$V(x) = \frac{1}{2}kx^2$$

The total energy E in the ground state is:

$$E = \frac{p^2}{2m} + \frac{1}{2}kx^2$$

Using the uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$, we set $\Delta p \approx p$ and $\Delta x \approx x$:

$$x \cdot p \geq \frac{\hbar}{2} \Rightarrow p \geq \frac{\hbar}{2x}$$

Substituting into the energy expression:

$$E \geq \frac{(\hbar/2x)^2}{2m} + \frac{1}{2}kx^2$$

Minimizing E with respect to x:

$$E = \frac{\hbar^2}{8mx^2} + \frac{1}{2}kx^2$$

Setting the derivative $\frac{dE}{dx} = 0$:

$$-\frac{\hbar^2}{4mx^3} + kx = 0 \Rightarrow x^4 = \frac{\hbar^2}{4mk} \Rightarrow x^2 = \frac{\hbar}{2\sqrt{mk}}$$

Substituting x^2 back into E:

$$E = \frac{\hbar^2}{8m \cdot \frac{\hbar}{2\sqrt{mk}}} + \frac{1}{2}k \cdot \frac{\hbar}{2\sqrt{mk}}$$

$$E = \frac{\hbar\sqrt{k/m}}{4} + \frac{\hbar\sqrt{k/m}}{4} = \frac{\hbar\omega}{2}$$

where $\omega = \sqrt{k/m}$.

Conclusion: The ground state energy of a linear harmonic oscillator is $\frac{\hbar\omega}{2}$, demonstrating the **zero-point energy due to quantum fluctuations**. Because of the zero-point energy, the position and momentum of the oscillator in the ground state are not fixed (as they would be in a classical oscillator), but have a small range of variance, in accordance with the Heisenberg uncertainty principle.

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5 In a series of experiments on the determination of the mass of a certain elementary particle, the results showed a variation of $\pm 20m_e$, where m_e is the electron mass. Estimate the lifetime of the particle. (2013)

Introduction: The uncertainty principle, formulated by Werner Heisenberg, states that two conjugate pair(which do not commute) in quantum mechanics can never be precisely measured simultaneously. In this case it states that energy and time cant be exactly determined simultaneously.

Solution: Given the mass uncertainty $\Delta m = \pm 20m_e$, where m_e is the electron mass, we use the energy-time uncertainty principle:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

The energy uncertainty ΔE can be related to the mass uncertainty Δm through $E = mc^2$:

$$\Delta E = \Delta mc^2$$

Substituting $\Delta m = 40m_e$:

$$\Delta E = 40m_e c^2$$

Using the uncertainty principle:

$$40m_e c^2 \Delta t \geq \frac{\hbar}{2}$$

$$\Delta t \geq \frac{\hbar}{2 \cdot 40m_e c^2}$$

$$\Delta t \geq \frac{\hbar}{80m_e c^2}$$

Given $\hbar \approx 1.054 \times 10^{-34}$ Js and $m_e c^2 \approx 8.187 \times 10^{-14}$ J:

$$\Delta t \geq \frac{1.054 \times 10^{-34}}{80 \times 8.187 \times 10^{-14}}$$

$$\Delta t \geq 1.61 \times 10^{-23}\,\text{s}$$

Conclusion: The estimated lifetime of the particle, based on its mass uncertainty, is 1.61×10^{-23} s. **It decays via a week force into a nucleon and a pion which highlights the precision required in high-energy physics experiments and the stability of the Lambda particle**.

6 Find the de Broglie wavelength of neutron and electron with kinetic energy 500 eV. (2014)

Introduction: The de Broglie wavelength is a fundamental concept in quantum mechanics, introduced by Louis de Broglie. It describes the wave-like behavior of particles and is inversely proportional to their momentum.

Solution: For a particle, the de Broglie wavelength λ is given by:

$$\lambda = \frac{h}{p}$$

where h is Planck's constant and p is the momentum of the particle.

(i) A neutron with kinetic energy of 500 eV: The kinetic energy E_k is related to the momentum p by:

$$E_k = \frac{p^2}{2m}$$

Solving for p:

$$p = \sqrt{2mE_k}$$

Substituting $E_k = 500\,\text{eV}$ and $1\,\text{eV} = 1.602 \times 10^{-19}\,\text{J}$:

$$E_k = 500 \times 1.602 \times 10^{-19}\,\text{J}$$

$$p = \sqrt{2 \times 1.675 \times 10^{-27}\,\text{kg} \times 500 \times 1.602 \times 10^{-19}\,\text{J}}$$

$$p = \sqrt{2863.35 \times 10^{-46}}\,\text{kg} \cdot \text{m/s}$$

$$p \approx 5.3 \times 10^{-22}\,\text{kg} \cdot \text{m/s}$$

The de Broglie wavelength is:

$$\lambda = \frac{6.626 \times 10^{-34}\,\text{Js}}{5.3 \times 10^{-22}\,\text{kg} \cdot \text{m/s}}$$

$$\lambda \approx 1.28 \times 10^{-12}\,\text{m}$$

(ii) An electron with kinetic energy of 500 eV: The kinetic energy E_k is related to the momentum p by:

$$E_k = \frac{p^2}{2m_e}$$

Solving for p:

$$p = \sqrt{2m_eE_k}$$

Substituting $E_k = 500\,\text{eV}$ and $1\,\text{eV} = 1.602 \times 10^{-19}\,\text{J}$:

$$E_k = 500 \times 1.602 \times 10^{-19}\,\text{J}$$

$$p = \sqrt{2 \times 9.11 \times 10^{-31}\,\text{kg} \times 500 \times 1.602 \times 10^{-19}\,\text{J}}$$

$$p = \sqrt{1.457 \times 10^{-46}}\,\text{kg} \cdot \text{m/s}$$

$$p \approx 1.21 \times 10^{-23}\,\text{kg} \cdot \text{m/s}$$

The de Broglie wavelength is:

$$\lambda = \frac{6.626 \times 10^{-34}\,\text{Js}}{1.21 \times 10^{-23}\,\text{kg} \cdot \text{m/s}}$$

$$\lambda \approx 5.48 \times 10^{-11}\ \mathrm{m}$$

Conclusion: The de Broglie wavelength of a neutron with kinetic energy of 500 eV is approximately 1.28×10^{-12} m, while that of an electron with the same kinetic energy is approximately 5.48×10^{-11} m. These results illustrate the wave-particle duality of matter, with significant differences in wavelengths due to the mass disparity between neutrons and electrons.

7 The mean life of Lambda (Λ^0) particle is 2.6×10^{-10} s. What will be the uncertainty in the determination of its mass in eV? (2014)

Introduction: The uncertainty principle, formulated by Werner Heisenberg, states that two conjugate pair (which do not commute) in quantum mechanics can never be precisely measured simultaneously. In this case it states that energy and time cant be exactly determined simultaneously

Solution: Given the mean life of Lambda particle (Λ^0) is 2.6×10^{-10} s, we use the energy-time uncertainty principle:

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

Rewriting in terms of mass uncertainty:

$$\Delta mc^2 \cdot \Delta t \geq \frac{\hbar}{2}$$

$$\Delta m \geq \frac{\hbar}{2c^2 \Delta t}$$

Given $\Delta t = 2.6 \times 10^{-10}$ s, $\hbar = 1.054 \times 10^{-34}$ Js, and $c = 3 \times 10^8$ m/s:

$$\Delta mc^2 \geq \frac{1.054 \times 10^{-34}}{2 \times 2.6 \times 10^{-10}}$$

$$\Delta mc^2 \geq \frac{1.054 \times 10^{-34}}{5.2 \times 10^{-10}}$$

$$\Delta E \geq 0.202 \times 10^{-24}\,\text{J}$$

Converting to energy using $1\,\text{eV} = 1.602 \times 10^{-19}$ J:

$$\Delta E \approx 0.202 \times 10^{-24}$$

$$\Delta E \approx 2.025 \times 10^{-25}\,\text{J}$$

$$\Delta E \approx \frac{2.025 \times 10^{-25}}{1.602 \times 10^{-19}}\,\text{eV}$$

$$\Delta E \approx 1.26 \times 10^{-6}\,\text{eV}$$

Conclusion: The uncertainty in the mass determination of the Lambda particle is approximately 1.26×10^{-6} eV. **It decays via a week force into a nucleon and a pion which highlights the precision required in high-energy physics experiments and the stability of the Lambda particle**.

8 Find the energy, momentum and wavelength of photon emitted by a hydrogen atom making a direct transition from an excited state with $n = 10$ to the ground state. Also find the recoil speed of the hydrogen atom in this process. (2016)

Introduction: Whenever an electron makes a transition from higher energy level to lower energy level it radiates energy in forms of quanta. The transited photon had a momentum and corresponding wavelenght assosiated with it.

Solution: For an electron transitioning from $n = 10$ to the ground state ($n = 1$), the energy difference is given by:

$$E_n = -13.6\frac{1}{n^2}\,\text{eV}$$

The energy of the photon emitted:

$$\Delta E = E_1 - E_{10}$$

$$E_1 = -13.6\,\text{eV}, \quad E_{10} = -13.6\frac{1}{10^2} = -0.136\,\text{eV}$$

$$\Delta E = -0.136\,\text{eV} - (-13.6\,\text{eV})$$

$$\Delta E = 13.464\,\text{eV}$$

The wavelength λ of the emitted photon:

$$E = \frac{hc}{\lambda}$$

$$\lambda = \frac{hc}{\Delta E}$$

Given $h = 6.626 \times 10^{-34}\,\text{Js}$, $c = 3 \times 10^8\,\text{m/s}$, and $\Delta E = 13.464 \times 1.602 \times 10^{-19}\,\text{J}$:

$$\lambda = \frac{6.626 \times 10^{-34} \times 3 \times 10^8}{13.464 \times 1.602 \times 10^{-19}}$$

$$\lambda \approx 9.13 \times 10^{-8}\,\text{m} \approx 91.3\,\text{nm}$$

To find the recoil speed v_r of the hydrogen atom:

$$p = \frac{E}{c} = \frac{13.464 \times 1.602 \times 10^{-19}}{3 \times 10^8} \approx 7.19 \times 10^{-27}\,\text{kg}\cdot\text{m/s}$$

Using momentum conservation, $p = Mv_r$:

$$v_r = \frac{p}{M}$$

Given $M \approx 1.67 \times 10^{-27}\,\text{kg}$:

$$v_r = \frac{7.19 \times 10^{-27}}{1.67 \times 10^{-27}}$$

$$v_r \approx 4.3\,\text{m/s}$$

Conclusion: The photon emitted in the hydrogen atom transition has a wavelength of 91.3 nm, and the recoil speed of the hydrogen atom is approximately 4.3 m/s. **These results illustrate the principles of energy quantization and conservation in atomic transitions, with applications in spectroscopy and quantum mechanics**.

9 An electron is confined to move between two rigid walls separated by 10^{-9} m. Compute the de Broglie wavelengths representing the first three allowed energy states of the electron and the corresponding energies. (2016)

Introduction: The de Broglie wavelength is a fundamental concept in quantum mechanics which describes the wave nature of particles. According to de Broglie's hypothesis, every moving particle or object has an associated wave. The wavelength is inversely proportional to its momentum. This concept was historically pivotal in the development of quantum mechanics.

Solution: To solve for the de Broglie wavelengths and the corresponding energies, we will use the particle in a box model. Here, the electron is confined in a one-dimensional potential well of width $L = 10^{-9}$ m.

The energy levels for a particle in a box are given by:

$$E_n = \frac{n^2 h^2}{8mL^2}$$

where: - n is the principal quantum number (1, 2, 3, ...), - h is Planck's constant, 6.626×10^{-34} Js, - m is the mass of the electron, 9.109×10^{-31} kg, - L is the width of the box.

Let's compute the first three energy levels.

For $n = 1$:

$$E_1 = \frac{1^2 \times (6.626 \times 10^{-34})^2}{8 \times 9.109 \times 10^{-31} \times (10^{-9})^2}$$

$$E_1 = \frac{6.626^2 \times 10^{-68}}{8 \times 9.109 \times 10^{-31} \times 10^{-18}}$$

$$E_1 = \frac{43.95 \times 10^{-68}}{7.287 \times 10^{-48}}$$

$$E_1 = 6.03 \times 10^{-20} \text{ J}$$

For $n = 2$:

$$E_2 = \frac{4 \times (6.626 \times 10^{-34})^2}{8 \times 9.109 \times 10^{-31} \times (10^{-9})^2}$$

$$E_2 = 4 \times E_1$$

$$E_2 = 4 \times 6.03 \times 10^{-20}$$

$$E_2 = 2.41 \times 10^{-19} \text{ J}$$

For $n = 3$:

$$E_3 = \frac{9 \times (6.626 \times 10^{-34})^2}{8 \times 9.109 \times 10^{-31} \times (10^{-9})^2}$$

$$E_3 = 9 \times E_1$$

$$E_3 = 9 \times 6.03 \times 10^{-20}$$

$$E_3 = 5.43 \times 10^{-19} \text{ J}$$

Now, the de Broglie wavelength is given by:

$$\lambda_n = \frac{h}{p_n}$$

where p_n is the momentum of the electron in the n-th energy state. For a particle in a box, the momentum is given by:

$$p_n = \sqrt{2mE_n}$$

So, for the first three states: For $n = 1$:

$$p_1 = \sqrt{2 \times 9.109 \times 10^{-31} \times 6.03 \times 10^{-20}}$$

$$p_1 = 3.3 \times 10^{-25}\,\text{kg m/s}$$

$$\lambda_1 = \frac{6.626 \times 10^{-34}}{3.3 \times 10^{-25}}$$

$$\lambda_1 = 2.007 \times 10^{-9}\,\text{m}$$

For $n = 2$:

$$p_2 = \sqrt{2 \times 9.109 \times 10^{-31} \times 2.41 \times 10^{-19}}$$

$$p_2 = \sqrt{4.38 \times 10^{-49}}$$

$$p_2 = 6.6 \times 10^{-25}\,\text{kg m/s}$$

$$\lambda_2 = \frac{6.626 \times 10^{-34}}{6.6 \times 10^{-25}}$$

$$\lambda_2 = \ 10^{-9}\,\text{m}$$

For $n = 3$:

$$p_3 = \sqrt{2 \times 9.109 \times 10^{-31} \times 5.43 \times 10^{-19}}$$

$$p_3 = \sqrt{9.89 \times 10^{-49}}$$

$$p_3 = 9.94 \times 10^{-25}\,\text{kg m/s}$$

$$\lambda_3 = \frac{6.626 \times 10^{-34}}{9.94 \times 10^{-25}}$$

$$\lambda_3 = 6.66 \times 10^{-10}\,\text{m}$$

Wavefunctions and Energy Levels:

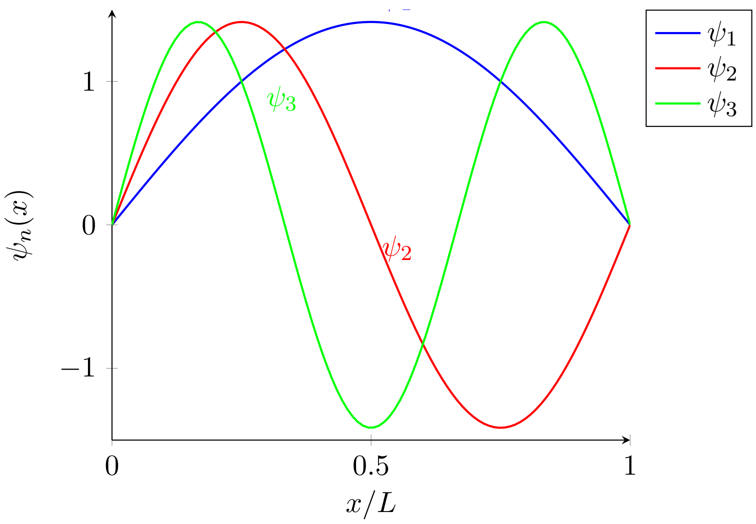

Figure 2: Wavefunctions for the first three energy states

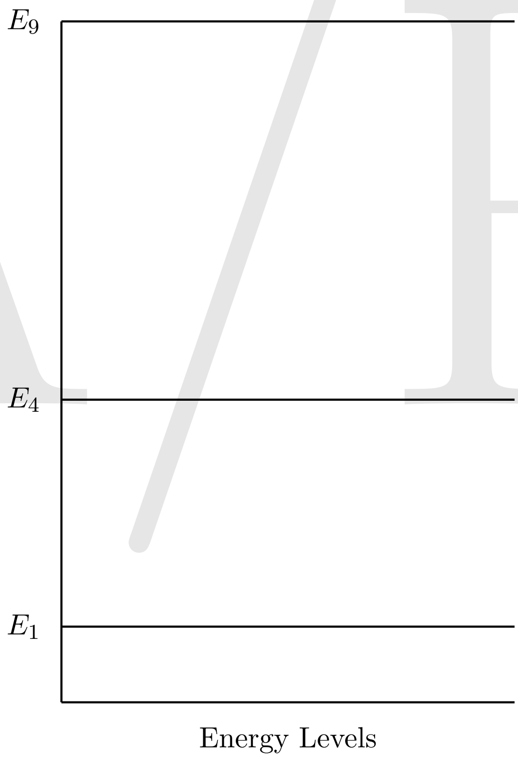

Figure 3: Energy levels for the first three states

Conclusion: These values reflect the quantized nature of energy levels in a confined system, significant in fields like **quantum computing and semiconductor physics**.

10 A typical atomic radius is about 5×10^{-15} m and the energy of β-particle emitted from a nucleus is at most of the order of 1 MeV. Prove on the basis of uncertainty principle that the electrons are not present in nuclei. (2016)

Introduction: The uncertainty principle, formulated by Werner Heisenberg, states that two conjugate pair (which do not commute) in quantum mechanics can never be precisely measured simultaneously. In this case it states that energy and time cant be exactly determined simultaneously..

Solution:

The Heisenberg Uncertainty Principle is given by:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

where Δx is the uncertainty in position and Δp is the uncertainty in momentum. Here, $\hbar$ is the reduced Planck's constant, $\hbar = \frac{h}{2\pi} \approx 1.055 \times 10^{-34}$ Js.

For an electron confined within a nucleus of radius $R \approx 5 \times 10^{-15}$ m, the uncertainty in position Δx is approximately the size of the nucleus:

$$\Delta x \approx 5 \times 10^{-15} \text{ m}$$

The uncertainty in momentum Δp can be found using the uncertainty principle:

$$\Delta p \geq \frac{\hbar}{2\Delta x}$$

Substituting the values:

$$\Delta p \geq \frac{1.055 \times 10^{-34}}{2 \times 5 \times 10^{-15}}$$
$$\Delta p \geq \frac{1.055 \times 10^{-34}}{10 \times 10^{-15}}$$
$$\Delta p \geq 1.055 \times 10^{-20} \text{ kg m/s}$$

The kinetic energy E of an electron can be related to its momentum p by the non-relativistic formula:

$$E = \frac{p^2}{2m}$$

where m is the mass of the electron, $m \approx 9.109 \times 10^{-31}$ kg. Using Δp for p:

$$E \geq \frac{(1.055 \times 10^{-20})^2}{2 \times 9.109 \times 10^{-31}}$$
$$E \geq \frac{1.113 \times 10^{-40}}{1.822 \times 10^{-30}}$$

$$E \geq 6.11 \times 10^{-11} \text{ J}$$

Converting this energy into electron volts ($1 \text{ eV} = 1.602 \times 10^{-19}$ J):

$$E \geq \frac{6.11 \times 10^{-11}}{1.602 \times 10^{-19}} \text{ eV}$$
$$E \geq 3.81 \times 10^{8} \text{ eV}$$
$$E \geq 381 \text{ MeV}$$

Conclusion: The minimum energy of an electron confined within a nucleus, according to the uncertainty principle, is approximately 381 MeV. This is significantly higher than the typical energy of β-particles emitted from a nucleus, which is about 1 MeV. Thus, electrons cannot be present in the nucleus as their confinement would require them to possess unreasonably high energy, inconsistent with observed nuclear phenomena.