UPSC PHYSICS PYQ SOLUTION

Quantum Mechanics - Part 2

Contents

11 A beam 4.0 keV electrons from a source is incident on a target 50.0 cm away. Find the radius of the electron beam spot due to Heisenberg's uncertainty principle. (2017)

Introduction: The uncertainty principle, formulated by Werner Heisenberg, states that the position and momentum of a particle cannot be simultaneously determined with arbitrary precision. For an electron beam, this principle limits how tightly the beam can be focused, leading to a minimum spot size on the target. And additionally it could be interpreted as a beam of electrons which is moving along a specified direction and it encounters a diaphragm with a slit and for that reason electrons under go diffraction.

Solution:

Given:

- Energy of electrons, $E = 4.0$ keV $= 4.0 \times 10^3 \times 1.602 \times 10^{-19}$ J $= 6.408 \times 10^{-16}$ J
- Distance to target, $L = 50.0$ cm $= 0.50$ m

Verification: Check if non-relativistic approximation is valid

The electron velocity is:

$$v = \frac{p}{m} = \frac{3.42 \times 10^{-23}}{9.109 \times 10^{-31}} = 3.75 \times 10^7 \text{ m/s}$$

Since $v/c = 3.75 \times 10^7/(3 \times 10^8) = 0.125 < 0.3$, the non-relativistic approximation is reasonable.

For non-relativistic electrons, the kinetic energy E is related to momentum by:

$$E = \frac{p^2}{2m}$$

where $m = 9.109 \times 10^{-31}$ kg is the electron mass.

Solving for momentum:

$$p = \sqrt{2mE}$$

Substituting values:

$$p = \sqrt{2 \times 9.109 \times 10^{-31} \times 6.408 \times 10^{-16}}$$

$$p = \sqrt{1.167 \times 10^{-45}} = 3.42 \times 10^{-23} \text{ kgm/s}$$

The uncertainty principle states:

$$\Delta x \cdot \Delta p \geq \frac{\hbar}{2}$$

where $\hbar = 1.055 \times 10^{-34}$ Js.

For minimum uncertainty (equality case):

$$\Delta x = \frac{\hbar}{2\Delta p}$$

If the initial beam has an uncertainty in transverse position Δx_0, then by the uncertainty principle, it must have a corresponding uncertainty in transverse momentum:

$$\Delta p_\perp \geq \frac{\hbar}{2\Delta x_0}$$

This transverse momentum uncertainty causes the beam to spread as it travels. The angular divergence is:

$$\theta \approx \frac{\Delta p_\perp}{p} = \frac{\hbar}{2\Delta x_0 \cdot p}$$

As the beam travels distance L, the radius of the spot becomes:

$$r = \Delta x_0 + L \cdot \theta = \Delta x_0 + L \cdot \frac{\hbar}{2\Delta x_0 \cdot p}$$

To minimize the spot size (as we want to ascertain the spot to be definite or you might think the radius from center to the first minima of the diffraction pattern), we differentiate with respect to Δx_0 and set equal to zero:

$$\frac{dr}{d\Delta x_0} = 1 - \frac{L\hbar}{2(\Delta x_0)^2 p} = 0$$

This gives the optimal initial beam width:

$$\Delta x_0 = \sqrt{\frac{L\hbar}{2p}}$$

The minimum spot radius is:

$$r = r_{min} = \Delta x_0 + L \cdot \theta = \Delta x_0 + L \cdot \frac{\hbar}{2\Delta x_0 \cdot p}$$

$$\begin{aligned} r_{min} &= \sqrt{\frac{L\hbar}{2p}} + \frac{L\hbar}{2p \cdot \sqrt{\frac{L\hbar}{2p}}} \\ &= \sqrt{\frac{L\hbar}{2p}} + \sqrt{\frac{L\hbar}{2p}} \\ &= 2\sqrt{\frac{L\hbar}{2p}} \\ &= \sqrt{\frac{2L\hbar}{p}} \end{aligned}$$

Substituting the values:

$$\begin{aligned} r_{\text{min}} &= \sqrt{\frac{2L\hbar}{p}} \\ &= \sqrt{\frac{2 \cdot 0.50 \cdot 1.055 \times 10^{-34}}{3.42 \times 10^{-23}}} \\ &= \sqrt{3.08 \times 10^{-12}} \\ &= 1.76 \times 10^{-6}\,\text{m} \\ &\approx \boxed{1.8\,\mu\text{m}} \end{aligned}$$

Therefore:

$$r_{\text{min}} \approx 1.8 \times 10^{-6} \text{ m} = 1.8\mu\text{m}$$

Conclusion: The minimum radius of the electron beam spot on the target due to Heisenberg's uncertainty principle is approximately 1.8μm. This fundamental quantum mechanical limit demonstrates why electron microscopes and other high-precision electron beam instruments face ultimate resolution limits determined by the uncertainty principle.

12 Estimate the de Broglie wavelength of the electron orbiting in the first excited state of the hydrogen atom. (2017)

Introduction:

The de Broglie wavelength is a fundamental concept in quantum mechanics that describes the wave-like behavior of particles. Introduced by Louis de Broglie in 1924, it posits that any moving particle has an associated wavelength given by $\lambda = \frac{h}{p}$, where h is Planck's constant and p is the particle's momentum. This principle bridges classical and quantum mechanics, highlighting the wave-particle duality of matter.

Solution:

We can also determine the de Broglie wavelength of the electron using the energy of the first excited state of the hydrogen atom. The total energy of an electron in the n-th orbit is given by:

$$E_n = -\frac{13.6\,\text{eV}}{n^2}$$

For the first excited state ($n = 2$):

$$E_2 = -\frac{13.6\,\text{eV}}{2^2} = -\frac{13.6\,\text{eV}}{4} = -3.4\,\text{eV}$$

This energy is the sum of the kinetic and potential energies. In the Bohr model of the hydrogen atom, the kinetic energy (K.E.) is equal to the negative of the total energy:

$$\text{K.E.} = -E_2 = 3.4\,\text{eV}$$

To find the momentum p of the electron, we use the relation between kinetic energy and momentum:

$$\text{K.E.} = \frac{p^2}{2m}$$

Solving for p:

$$p = \sqrt{2m \cdot \text{K.E.}}$$

Converting the kinetic energy to joules:

$$3.4\,\text{eV} = 3.4 \times 1.602 \times 10^{-19}\,\text{J} = 5.447 \times 10^{-19}\,\text{J}$$

Now, substituting the mass of the electron $m = 9.109 \times 10^{-31}\,\text{kg}$:

$$p = \sqrt{2 \cdot 9.109 \times 10^{-31}\,\text{kg} \cdot 5.447 \times 10^{-19}\,\text{J}}$$

$$p = \sqrt{9.919 \times 10^{-49}\,\text{kg}^2 \cdot \text{m}^2 \cdot \text{s}^{-2}}$$

$$p \approx 9.96 \times 10^{-25}\,\text{kg m/s}$$

Finally, we calculate the de Broglie wavelength λ:

$$\lambda = \frac{h}{p} = \frac{6.626 \times 10^{-34}\,\text{Js}}{9.96 \times 10^{-25}\,\text{kg m/s}}$$

$$\lambda \approx 6.65 \times 10^{-10}\,\text{m}$$

$$\lambda \approx 0.665\,\text{nm}$$

Conclusion:

The de Broglie wavelength of the electron in the first excited state of the hydrogen atom is approximately 0.665 nm. This wavelength reflects the wave-particle duality of the electron, emphasizing its quantum mechanical nature. Such insights are crucial for understanding phenomena at atomic scales, including electron diffraction and the formation of atomic spectra.

13 Show that the mass and linear momentum of a quantum mechanical particle can be given by $m = \frac{h}{\lambda v}$ and $p = \frac{h}{\lambda}$, respectively, where h, λ and v are Planck's constant, wavelength, and velocity of the particle, respectively. Comment on the wave-particle duality from these relations. (2019)

Introduction:

The de Broglie hypothesis posits that every moving particle has an associated wavelength, bridging the gap between classical and quantum physics. This hypothesis, introduced by Louis de Broglie in 1924, demonstrates the wave-particle duality, a cornerstone of quantum mechanics.

Solution:

To show the given relations, we start from the de Broglie wavelength formula. The de Broglie wavelength λ is given by:

$$\lambda = \frac{h}{p}$$

where h is Planck's constant and p is the momentum of the particle.

1. Derivation of momentum: Given the de Broglie relation:

$$\lambda = \frac{h}{p}$$

Solving for p:

$$p = \frac{h}{\lambda}$$

2. Derivation of mass: We also know that momentum p is related to mass m and velocity v by:

$$p = mv$$

Substituting $p = \frac{h}{\lambda}$ from the de Broglie relation:

$$mv = \frac{h}{\lambda}$$

Solving for m:

$$m = \frac{h}{\lambda v}$$

These derivations show that the mass and momentum of a quantum mechanical particle can be expressed in terms of Planck's constant, the particle's wavelength, and its velocity.

Comment on Wave-Particle Duality:

The derived relations $m = \frac{h}{\lambda v}$ and $p = \frac{h}{\lambda}$ underscore the wave-particle duality of matter. They reveal that the properties traditionally associated with particles (mass and momentum) can be described using wave characteristics (wavelength). This duality is fundamental in quantum mechanics, explaining phenomena such as electron diffraction and the quantization of atomic orbits.

Conclusion:

The expressions for mass and momentum derived from the de Broglie wavelength highlight the intrinsic connection between wave and particle properties in quantum mechanics. This wave-particle duality is crucial for understanding various quantum phenomena and has practical applications in fields like **electron microscopy and semiconductor technology**.

14 State and express mathematically the three uncertainty principles of Heisenberg. Highlight the physical significance of these principles in the development of Quantum Mechanics. (2019)

Introduction:

The Heisenberg uncertainty principle is a fundamental concept in quantum mechanics, formulated by Werner Heisenberg in 1927. It states that certain pairs of physical properties, known as **conjugate pairs, cannot both be known to arbitrary precision simultaneously**. This principle is mathematically expressed using the commutator of these conjugate pairs, which is non-zero.

Solution:

Heisenberg's uncertainty principle can be expressed mathematically for three different pairs of conjugate variables:

1. **Position and Momentum:**

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

The commutator for position $\hat{x}$ and momentum $\hat{p}$ is $[\hat{x}, \hat{p}] = i\hbar$. Since this commutator is not zero, it implies that position and momentum cannot be simultaneously determined with arbitrary precision.

2. **Energy and Time:**

$$\Delta E \Delta t \geq \frac{\hbar}{2}$$

The commutator for energy $\hat{E}$ and time $\hat{t}$ is $[\hat{E}, \hat{t}] = i\hbar$. This non-zero commutator signifies that energy and time cannot both be precisely measured at the same time.

3. **Angular Position and Angular Momentum:**

$$\Delta \theta \Delta L \geq \frac{\hbar}{2}$$

The commutator for angular position $\hat{\theta}$ and angular momentum $\hat{L}$ is $[\hat{\theta}, \hat{L}] = i\hbar$. Again, the non-zero commutator indicates that angular position and angular momentum cannot be simultaneously determined with arbitrary precision.

These inequalities show that increasing the precision in measuring one quantity leads to increased uncertainty in the conjugate quantity.

Physical Significance:

The uncertainty principles have several significant implications:

1. **Limits of Measurement:** They set fundamental limits on the precision of measurements, illustrating that there is a limit to how precisely we can simultaneously know certain pairs of properties of a quantum system.

2. **Wave-Particle Duality:** These principles highlight the wave-particle duality of matter, emphasizing that particles exhibit both wave-like and particle-like properties, depending on the measurement context.

3. **Quantum Behavior:** The principles help explain why atoms do not collapse, as electrons cannot have both a well-defined position and momentum. This results in stable atomic structures.

4. **Quantum Fluctuations:** In fields such as quantum field theory, the energy-time uncertainty principle is crucial for understanding quantum fluctuations and the creation of particle-antiparticle pairs.

Conclusion:

Heisenberg's uncertainty principles are cornerstones of quantum mechanics, fundamentally altering our understanding of measurement and the behavior of particles at microscopic scales. They underscore the intrinsic limitations of classical concepts when applied to quantum systems and have wide-ranging applications in technology and theoretical physics, such as in the development of quantum computers and the study of fundamental particles.

15 For a free quantum mechanical particle under the influence of a one-dimensional potential, show that the energy is quantized in discrete fashion. How do these energy values differ from those of a linear harmonic oscillator? (2019)

Introduction:

Quantum mechanics reveals that particles can only occupy certain discrete energy levels, a phenomenon known as quantization. This concept was developed in the early 20th century by scientists like Planck, Bohr, and Schrödinger. Quantization arises due to boundary conditions and the wave nature of particles.

Solution:

To show the quantization of energy, we consider a particle in a one-dimensional potential well (infinite potential well) of width L.

The time-independent Schrödinger equation is:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} = E\psi(x)$$

For a particle in an infinite potential well, the potential $V(x)$ is:

$$V(x) = \begin{cases} 0 & \text{for } 0 < x < L \\ \infty & \text{otherwise} \end{cases}$$

Inside the well, the Schrödinger equation simplifies to:

$$\frac{d^2\psi(x)}{dx^2} + \frac{2mE}{\hbar^2}\psi(x) = 0$$

Let $k^2 = \frac{2mE}{\hbar^2}$, then:

$$\frac{d^2\psi(x)}{dx^2} + k^2\psi(x) = 0$$

The general solution to this differential equation is:

$$\psi(x) = A\sin(kx) + B\cos(kx)$$

Applying boundary conditions $\psi(0) = 0$ and $\psi(L) = 0$: 1. At $x = 0$:

$$\psi(0) = A\sin(0) + B\cos(0) = B = 0$$

So, $\psi(x) = A\sin(kx)$.

2. At $x = L$:

$$\psi(L) = A\sin(kL) = 0$$

Since $A \neq 0$, we must have $\sin(kL) = 0$.

Thus, $kL = n\pi$, where n is an integer ($n = 1, 2, 3, \ldots$).

So, $k = \frac{n\pi}{L}$.

The energy levels are given by:

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m}\left(\frac{n\pi}{L}\right)^2 = \frac{n^2\pi^2\hbar^2}{2mL^2}$$

Hence, the energy is quantized and the allowed energies are:

$$E_n = \frac{n^2\pi^2\hbar^2}{2mL^2} \quad \text{for } n = 1, 2, 3, \ldots$$

Comparison with Linear Harmonic Oscillator:

For a linear harmonic oscillator, the energy levels are given by:

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega \quad \text{for } n = 0, 1, 2, \ldots$$

The key differences are:

1. The energy levels for the particle in a potential well are proportional to n^2, while for the harmonic oscillator they are proportional to $n + \frac{1}{2}$.

2. The spacing between energy levels in the potential well increases with n, whereas for the harmonic oscillator, the spacing between adjacent energy levels is constant ($\hbar\omega$)

3. There is no zero point energy in case of 1D infinite well as is the case with harmonic oscillator.

4. There is fixed boundary condition in case of infinite well i.e 0 to L but for harmonic oscillator there is a restoring force but no fixed spatial boundary condition due to which they have different energy level properties.

Conclusion:

The quantization of energy in a one-dimensional potential well demonstrates how boundary conditions lead to discrete energy levels. This concept is foundational in quantum mechanics, affecting phenomena like electron configurations in atoms and the behavior of particles in confined spaces. The comparison with the linear harmonic oscillator highlights the diversity in quantum systems, each with unique energy quantization characteristics.

16 Using the uncertainty principle $\Delta x \Delta p \geq \hbar/2$, estimate the ground state energy of a harmonic oscillator. (2020)

Introduction: The uncertainty principle, formulated by Werner Heisenberg in 1927, states that it is impossible to simultaneously determine the exact position and momentum of a particle. This principle is fundamental to quantum mechanics and impacts the behavior of quantum systems such as the harmonic oscillator.

Solution:

The ground state energy of a harmonic oscillator can be estimated using the uncertainty principle. For a harmonic oscillator, the potential energy $V(x) = \frac{1}{2}m\omega^2 x^2$ and the kinetic energy $T(p) = \frac{p^2}{2m}$.

Using the uncertainty principle $\Delta x \Delta p \geq \frac{\hbar}{2}$:

$$\Delta p \approx \frac{\hbar}{2\Delta x}$$

The total energy E is given by the sum of kinetic and potential energies. Assuming Δx is of the order of the position uncertainty and Δp is of the order of the momentum uncertainty:

$$E \approx \frac{(\Delta p)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$$

Substituting Δp:

$$E \approx \frac{\left(\frac{\hbar}{2\Delta x}\right)^2}{2m} + \frac{1}{2}m\omega^2(\Delta x)^2$$

$$E \approx \frac{\hbar^2}{8m(\Delta x)^2} + \frac{1}{2}m\omega^2(\Delta x)^2$$

Minimizing this energy with respect to Δx, we set the derivative with respect to Δx to zero:

$$\frac{dE}{d(\Delta x)} = -\frac{\hbar^2}{4m(\Delta x)^3} + m\omega^2(\Delta x) = 0$$

$$-\frac{\hbar^2}{4m(\Delta x)^3} + m\omega^2(\Delta x) = 0$$

$$\frac{\hbar^2}{4m^2(\Delta x)^4} = \omega^2$$

$$(\Delta x)^4 = \frac{\hbar^2}{4m^2\omega^2}$$

$$(\Delta x)^2 = \frac{\hbar}{2m\omega}$$

$$\Delta x = \sqrt{\frac{\hbar}{2m\omega}}$$

Substituting back into the energy expression:

$$E \approx \frac{\hbar^2}{8m}\left(\frac{2m\omega}{\hbar}\right) + \frac{1}{2}m\omega^2\left(\frac{\hbar}{2m\omega}\right)$$

$$E \approx \frac{\hbar\omega}{4} + \frac{\hbar\omega}{4} = \frac{\hbar\omega}{2}$$

Thus, the ground state energy is:

$$E_0 = \frac{\hbar\omega}{2}$$

Conclusion:

The uncertainty principle is crucial in understanding the limitations of measurements at the quantum level. The calculated ground state energy of a harmonic oscillator being $\frac{\hbar\omega}{2}$ signifies the zero-point energy, indicating that even at absolute zero, the oscillator retains quantum mechanical motion. This concept is widely applicable in fields like quantum field theory and low-temperature physics.

17 A blue lamp emits light of mean wavelength of 4500 Å. The rating of the lamp is 150 W and its 8% of the energy appears as light. How many photons are emitted per second by the lamp? (2020)

Introduction: Photon emission from light sources can be quantified using the energy-wavelength relationship for photons. This relationship is fundamental in quantum mechanics and is instrumental in understanding light sources.

Solution:

First, convert the wavelength from angstroms to meters:

$$\lambda = 4500\,\text{Å} = 4500 \times 10^{-10}\,\text{m} = 4.5 \times 10^{-7}\,\text{m}$$

The energy of one photon E is given by:

$$E = \frac{hc}{\lambda}$$

where h is Planck's constant (6.626×10^{-34} Js) and c is the speed of light (3×10^{8} m/s).

$$E = \frac{6.626 \times 10^{-34} \times 3 \times 10^{8}}{4.5 \times 10^{-7}} = \frac{19.878 \times 10^{-26}}{4.5 \times 10^{-7}} = 4.417 \times 10^{-19}\,\text{J}$$

The power output of the lamp as light is 8% of 150 W:

$$P = 0.08 \times 150 = 12\,\text{W}$$

The number of photons emitted per second N is given by:

$$N = \frac{P}{E} = \frac{12}{4.417 \times 10^{-19}} = 2.717 \times 10^{19}$$

Thus, the number of photons emitted per second by the lamp is:

$$N \approx 2.72 \times 10^{19}\,\text{photons/s}$$

Conclusion:

Photon emission quantification allows for precise control and application in various technologies such as lasers, LEDs, and other optical devices. The calculation of photon emission rate is crucial in designing efficient lighting systems and understanding the energy efficiency of light sources.

18 Consider a Hermitian operator A with property $A^3 = 1$. Show that $A = 1$. (2020)

Introduction: Hermitian operators play a critical role in quantum mechanics, especially because their eigenvalues are real. The problem explores the properties of a specific Hermitian operator.

Solution:

Given that A is a Hermitian operator, all its eigenvalues are real. Let λ be an eigenvalue of A with an eigenvector $|\psi\rangle$, i.e.,

$$A|\psi\rangle = \lambda|\psi\rangle$$

Given $A^3 = 1$,

$$A^3|\psi\rangle = 1|\psi\rangle$$

$$\lambda^3|\psi\rangle = |\psi\rangle$$

$$\lambda^3 = 1$$

The real solutions to $\lambda^3 = 1$ are $\lambda = 1$. Hence, the only eigenvalue of A is 1.

Since A is Hermitian and all its eigenvalues are 1, we can write:

$$A = I$$

Therefore,

$$A = 1$$

Conclusion:

Hermitian operators are fundamental in ensuring that measurements in quantum mechanics yield real values. The result demonstrates the specific behavior of a Hermitian operator with a given property, reinforcing the concept that such operators have real eigenvalues, which in this case leads to a unique solution. This concept has applications in quantum computing and spectral theory.

19 Find the uncertainty in the momentum of a particle when its position is determined within 0.02 cm. Find also the uncertainty in the velocity of an electron and α-particle respectively when they are located within 15×10^{-8}cm. (2020)

Introduction:

The Heisenberg uncertainty principle states that it is impossible to simultaneously determine the exact position and momentum of a particle. This principle is fundamental to quantum mechanics and provides limits on how precisely we can measure these quantities.

Solution:

The uncertainty principle is given by:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

First, let's find the uncertainty in momentum Δp when the position $\Delta x = 0.02\,\text{cm} = 0.02 \times 10^{-2}\,\text{m}$.

$$\Delta p \geq \frac{\hbar}{2\Delta x}$$

Using $\hbar = 1.054 \times 10^{-34}$ Js,

$$\Delta p \geq \frac{1.054 \times 10^{-34}}{2 \times 0.02 \times 10^{-2}} \approx 2.635 \times 10^{-32}\,\text{kg m/s}$$

Now, let's find the uncertainty in velocity Δv for an electron and an α-particle when the position $\Delta x = 15 \times 10^{-8}\,\text{cm} = 15 \times 10^{-10}\,\text{m}$.

For an electron with mass $m_e = 9.11 \times 10^{-31}\,\text{kg}$,

$$\Delta p \geq \frac{\hbar}{2\Delta x} = \frac{1.054 \times 10^{-34}}{2 \times 15 \times 10^{-10}} \approx 3.513 \times 10^{-26}\,\text{kg m/s}$$

$$\Delta v_e \geq \frac{\Delta p}{m_e} = \frac{3.513 \times 10^{-26}}{9.11 \times 10^{-31}} \approx 3.86 \times 10^{4}\,\text{m/s}$$

For an α-particle with mass $m_\alpha = 4 \times 1.66 \times 10^{-27}\,\text{kg} = 6.64 \times 10^{-27}\,\text{kg}$,

$$\Delta v_\alpha \geq \frac{\Delta p}{m_\alpha} = \frac{3.513 \times 10^{-26}}{6.64 \times 10^{-27}} \approx 5.29 \times 10^{0}\,\text{m/s}$$

Conclusion: The uncertainty in the momentum of a particle when its position is determined within $0.02\,\text{cm}$ is approximately 2.635×10^{-32} kg m/s. For an electron and an α-particle located within $15 \times 10^{-8}\,\text{cm}$, the uncertainties in their velocities are approximately $3.86 \times 10^{4}\,\text{m/s}$ and $5.29\,\text{m/s}$, respectively. This illustrates the significant impact of particle mass on the uncertainty in velocity, highlighting the precision limitations inherent in quantum measurements.

20 A particle of rest mass m_0 has a kinetic energy K, show that its de Broglie wavelength is given by $\lambda = \frac{hc}{\sqrt{K(K+2m_0c^2)}}$. Hence calculate the wavelength of an electron of kinetic energy 2 MeV. What will be the value of λ if $K \ll m_0c^2$? (2020)

Introduction: The de Broglie wavelength relates a particle's momentum to its wavelength, an essential concept in quantum mechanics introduced by Louis de Broglie in 1924. This concept is pivotal in understanding wave-particle duality.

Solution:

The total energy E of a particle is given by:

$$E = K + m_0c^2$$

The momentum p of the particle is related to its energy and mass by the relation:

$$E^2 = (pc)^2 + (m_0c^2)^2$$

Substituting $E = K + m_0c^2$,

$$(K + m_0c^2)^2 = (pc)^2 + (m_0c^2)^2$$

$$K^2 + 2Km_0c^2 + (m_0c^2)^2 = (pc)^2 + (m_0c^2)^2$$

Subtracting $(m_0c^2)^2$ from both sides,

$$K^2 + 2Km_0c^2 = (pc)^2$$

$$p = \frac{\sqrt{K^2 + 2Km_0c^2}}{c}$$

The de Broglie wavelength λ is given by:

$$\lambda = \frac{h}{p} = \frac{h}{\frac{\sqrt{K^2+2Km_0c^2}}{c}} = \frac{hc}{\sqrt{K^2 + 2Km_0c^2}}$$

Thus, the de Broglie wavelength is:

$$\lambda = \frac{hc}{\sqrt{K(K + 2m_0c^2)}}$$

Next, let's calculate the wavelength of an electron with kinetic energy $K = 2\,\text{MeV}$.

First, convert the kinetic energy to joules:

$$K = 2\,\text{MeV} = 2 \times 10^6 \times 1.602 \times 10^{-13}\,\text{J} = 3.204 \times 10^{-13}\,\text{J}$$

For an electron, $m_0 = 9.11 \times 10^{-31}\,\text{kg}$ and $c = 3 \times 10^8\,\text{m/s}$.

Calculate m_0c^2:

$$m_0c^2 = 9.11 \times 10^{-31} \times (3 \times 10^8)^2\,\text{J} = 8.2 \times 10^{-14}\,\text{J}$$

Now, calculate λ:

$$\lambda = \frac{hc}{\sqrt{K(K + 2m_0c^2)}}$$

Using $h = 6.626 \times 10^{-34}\,\text{Js}$,

$$\lambda = \frac{1.988 \times 10^{-25}}{\sqrt{3.204 \times 10^{-13} \times 4.844 \times 10^{-13}}}$$

$$\lambda = \frac{1.988 \times 10^{-25}}{\sqrt{1.551 \times 10^{-25}}}$$

$$\lambda = \frac{1.988 \times 10^{-25}}{1.245 \times 10^{-12}} \approx 1.597 \times 10^{-13}\,\text{m}$$

For $K \ll m_0c^2$, $K + 2m_0c^2 \approx 2m_0c^2$,

$$\lambda = \frac{hc}{\sqrt{K(2m_0c^2)}} = \frac{hc}{\sqrt{2Km_0c^2}}$$

$$\lambda \approx \frac{h}{\sqrt{2m_0K}} \times \frac{c}{c} = \frac{h}{\sqrt{2m_0K}}$$

Conclusion: The derived expression for the de Broglie wavelength $\lambda = \frac{hc}{\sqrt{K(K+2m_0c^2)}}$ links a particle's kinetic energy to its wavelength, emphasizing the relationship between energy, momentum, and wavelength in quantum mechanics.

For an electron with kinetic energy of 2 MeV, the wavelength is approximately $1.597 \times 10^{-13}\,\text{m}$. When $K \ll m_0c^2$, the wavelength simplifies to $\lambda \approx \frac{h}{\sqrt{2m_0K}}$, highlighting the classical limit of the de Broglie wavelength.

This relation is significant in analyzing particle behavior at quantum scales, with applications in electron microscopy and particle physics.