UPSC PHYSICS PYQ SOLUTION

Waves and Optics - Part 1

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10 In a certain engine, a piston undergoes vertical SHM with an amplitude of $10\,\mathrm{cm}$. A washer rests on the top of the piston. As the motor is slowly speeded up, at what frequency will the washer no longer stay in contact with the piston? 19

1 In the propagation of longitudinal waves in a fluid contained in an infinitely long tube of cross-section A, show that $\rho = \rho_0 \left(1 - \frac{\partial \xi}{\partial x}\right)$, where ρ_0 is the equilibrium density, ρ is the disturbed density, and $\frac{\partial \xi}{\partial x}$ is the volume strain $\left(\left|\frac{\partial \xi}{\partial x}\right| \ll 1\right)$.

Introduction: In this problem, we examine the relationship between the equilibrium density ρ_0 of a fluid and its disturbed density ρ due to a longitudinal wave propagating through an infinitely long tube of cross-sectional area A. The displacement of fluid elements from equilibrium is represented by $\xi(x, t)$, where ξ is the displacement of a fluid particle initially at position x. The volume strain is given by $\dfrac{\partial \xi}{\partial x}$. The goal is to derive the expression:

$$\rho = \rho_0 \left(1 - \frac{\partial \xi}{\partial x}\right)$$

under the assumption that the strain is small, i.e., $\left|\dfrac{\partial \xi}{\partial x}\right| \ll 1$.

Solution:

Consider a small fluid element of initial length Δx in the undisturbed state. Its equilibrium volume is:

$$V_0 = A\Delta x$$

and its mass is:

$$m = \rho_0 V_0 = \rho_0 A \Delta x$$

Due to the passage of a longitudinal wave, let the displacement of the fluid at position x be $\xi(x)$ and at $x + \Delta x$ be $\xi(x + \Delta x)$. The new length of the element becomes:

$$\Delta x' = (x + \Delta x + \xi(x + \Delta x)) - (x + \xi(x)) = \Delta x + \xi(x + \Delta x) - \xi(x)$$

Expanding $\xi(x + \Delta x)$ in a Taylor series:

$$\xi(x + \Delta x) \approx \xi(x) + \Delta x \frac{\partial \xi}{\partial x}$$

Hence,

$$\Delta x' \approx \Delta x + \Delta x \frac{\partial \xi}{\partial x} = \Delta x \left(1 + \frac{\partial \xi}{\partial x}\right)$$

So, the new volume of the fluid element is:

$$V = A\Delta x' = A\Delta x \left(1 + \frac{\partial \xi}{\partial x}\right)$$

By conservation of mass, the mass of the fluid element remains constant, so the disturbed density is:

$$\rho = \frac{m}{V} = \frac{\rho_0 A \Delta x}{A\Delta x(1 + \frac{\partial \xi}{\partial x})} = \frac{\rho_0}{1 + \frac{\partial \xi}{\partial x}}$$

For small strain, $\left|\frac{\partial \xi}{\partial x}\right| \ll 1$, we use the binomial approximation:

$$\frac{1}{1+\frac{\partial \xi}{\partial x}} \approx 1 - \frac{\partial \xi}{\partial x}$$

Thus,

$$\rho \approx \rho_0 \left(1 - \frac{\partial \xi}{\partial x}\right)$$

Conclusion: We have shown that under the assumption of small volume strain, the disturbed density ρ of the fluid in a longitudinal wave is given by:

$$\rho = \rho_0 \left(1 - \frac{\partial \xi}{\partial x}\right)$$

This relation expresses how a local compression (negative $\frac{\partial \xi}{\partial x}$, where particles move closer together) increases the density above ρ_0, while a local rarefaction (positive $\frac{\partial \xi}{\partial x}$, where particles move apart) decreases the density below ρ_0.

2 Write down the one-dimensional harmonic oscillator differential equation under damping and its solution for the lightly damped condition, with the meanings of symbols. Determine the dependent energy in the lightly damped condition.

Differential Equation: The equation of motion for a damped harmonic oscillator is:

$$m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = 0$$

where:

- $x(t)$: Displacement as a function of time,
- m: Mass of the oscillator,
- c: Damping coefficient,
- k: Spring constant.

Rewritten Form: Define:

$$\omega_0 = \sqrt{\frac{k}{m}} \quad \text{(natural frequency)}, \quad \gamma = \frac{c}{2m} \quad \text{(damping parameter)}.$$

The equation becomes:

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = 0.$$

Solution for Light Damping ($\gamma < \omega_0$): Assume $x(t) = e^{rt}$, yielding the characteristic equation:

$$r^2 + 2\gamma r + \omega_0^2 = 0.$$

The roots are complex:

$$r = -\gamma \pm i\omega_d, \quad \omega_d = \sqrt{\omega_0^2 - \gamma^2}.$$

The general solution is:

$$x(t) = Ae^{-\gamma t}\cos(\omega_d t + \phi),$$

where:

- A: Initial amplitude (from $A = \sqrt{C_1^2 + C_2^2}$),
- ϕ: Phase constant (from $\tan\phi = -C_2/C_1$),
- ω_d: Damped angular frequency.

Energy in Lightly Damped Case: The total mechanical energy is:

$$E(t) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2.$$

Substitute $x(t)$ and $\dot{x}(t) = Ae^{-\gamma t}[-\gamma\cos(\omega_d t + \phi) - \omega_d\sin(\omega_d t + \phi)]$:

Using $\cos^2\theta + \sin^2\theta = 1$ and averaging over fast oscillations (since $\gamma \ll \omega_0$):

$$E(t) \approx \frac{1}{2}mA^2e^{-2\gamma t}(\omega_0^2).$$

Thus:

$$E(t) = E_0 e^{-2\gamma t}, \quad E_0 = \frac{1}{2} m A^2 \omega_0^2.$$

Conclusion: The displacement follows an exponentially decaying oscillation:

$$x(t) = A e^{-\gamma t} \cos(\omega_d t + \phi),$$

and the energy decays as:

$$E(t) = E_0 e^{-2\gamma t}.$$

This reflects energy dissipation at a rate 2γ.

3 Explain the physical significance of group velocity from the concept of phase velocity with relevant expressions.

Introduction: In wave mechanics, especially when dealing with wave packets or modulated waves, it is essential to distinguish between phase velocity and group velocity. Phase velocity describes the motion of individual wave crests, while group velocity corresponds to the propagation of the wave packet envelope that carries energy or information. This problem asks for the physical significance of group velocity in relation to phase velocity, including the relevant mathematical framework.

Solution: We begin by defining two types of velocities associated with waves:

- **Phase velocity** (v_p): This is the speed at which individual points of constant phase (like wave crests or troughs) propagate in a wave.
- **Group velocity** (v_g): This is the speed at which the overall shape or envelope of a wave packet (a localized group of waves) propagates. It represents the speed of energy and information transfer.

Consider the superposition of two harmonic waves with slightly different frequencies (ω_1, ω_2) and wave numbers (k_1, k_2):

$$y_1 = A\cos(k_1 x - \omega_1 t)$$

$$y_2 = A\cos(k_2 x - \omega_2 t)$$

Using the trigonometric identity $\cos A + \cos B = 2\cos\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$, their superposition ($y = y_1 + y_2$) yields:

$$y = 2A\cos\left(\frac{k_2 - k_1}{2}x - \frac{\omega_2 - \omega_1}{2}t\right)\cos\left(\frac{k_1 + k_2}{2}x - \frac{\omega_1 + \omega_2}{2}t\right)$$

Let $\Delta k = k_2 - k_1$, $\Delta\omega = \omega_2 - \omega_1$, $\bar{k} = \frac{k_1+k_2}{2}$, and $\bar{\omega} = \frac{\omega_1+\omega_2}{2}$. The equation can be rewritten as:

$$y = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos\left(\bar{k}x - \bar{\omega}t\right)$$

This equation describes a wave whose amplitude varies slowly in space and time. The first cosine term, $\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)$, represents the **envelope** of the wave packet, which has a much longer wavelength and period than the second cosine term, $\cos\left(\bar{k}x - \bar{\omega}t\right)$, which represents the **carrier wave**.

From the above superposition, we can define the velocities:

- **Phase velocity** (v_p) of the individual (carrier) waves:

$$v_p = \frac{\bar{\omega}}{\bar{k}}$$

- **Group velocity** (v_g) of the envelope: The speed of the envelope is determined by the term $\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x$. For a constant phase point on the envelope, we set the argument to a constant:

$$\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t = \text{constant}$$

Differentiating with respect to time, we get:

$$\frac{\Delta k}{2}\frac{dx}{dt} - \frac{\Delta\omega}{2} = 0$$

Thus, the velocity of the envelope is $\frac{dx}{dt} = \frac{\Delta\omega}{\Delta k}$. In the limit as $\Delta k \to 0$ (i.e., for a continuous distribution of wave numbers), the group velocity is defined as:

$$v_g = \lim_{\Delta k \to 0} \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk}$$

Physical Interpretation are as follows:

- **Energy and Information Transport**: The most significant physical interpretation of group velocity is that it represents the speed at which **energy** and **information** are transported by a wave. Unlike phase velocity, which can sometimes exceed the speed of light or even be negative, group velocity always adheres to the principles of causality and special relativity, meaning $v_g \le c$ (the speed of light in vacuum).

- **Dispersion**: In a **dispersive medium**, the phase velocity ($v_p = \omega/k$) depends on the frequency ω (or equivalently, the wave number k). This means that different frequency components of a wave packet travel at different phase velocities. As a result, the wave packet spreads out over time, and the group velocity differs from the phase velocity. Mathematically:

  $$v_g = \frac{d\omega}{dk} \neq v_p = \frac{\omega}{k}$$

  In a **non-dispersive medium**, ω is directly proportional to k ($\omega = vk$, where v is a constant). In this case, $d\omega/dk = v$, and $\omega/k = v$, so $v_g = v_p$.

- **Examples**:

  - **Deep water waves**: For deep water waves, the dispersion relation is $\omega = \sqrt{gk}$, where g is the acceleration due to gravity. Calculating the velocities:

    $$v_p = \frac{\omega}{k} = \frac{\sqrt{gk}}{k} = \sqrt{\frac{g}{k}}$$

    $$v_g = \frac{d\omega}{dk} = \frac{d}{dk}(\sqrt{gk}) = \frac{1}{2}\sqrt{\frac{g}{k}} = \frac{1}{2}v_p$$

    This means that in deep water, the energy of a wave travels at half the speed of its individual crests.

  - **Electromagnetic waves in dielectric media**: Light pulses in a dielectric medium also exhibit dispersion, where the group velocity determines the speed at which the light pulse (and thus information) propagates.

  - **Quantum wave packets**: In quantum mechanics, a particle is described by a wave packet. The group velocity of this wave packet corresponds to the classical velocity of the particle.

Conclusion: The group velocity, defined as $v_g = d\omega/dk$, is a fundamental concept in wave theory. It represents the speed at which the envelope of a wave packet propagates, signifying the rate of **energy** and **information** transfer. While phase velocity describes the motion of individual wave crests, group velocity is the physically meaningful velocity for observable phenomena. The distinction between v_g and v_p is crucial in **dispersive media**, where different frequency components travel at different speeds, leading to the spreading of wave packets. This concept finds widespread applications in fields such as optics, acoustics, quantum mechanics, and fluid dynamics.

4 Prove that the group velocity V_g of electromagnetic waves in a dispersive medium with refractive index $n(\lambda_0)$ at wavelength λ_0 is given by

$$V_g = \frac{c}{n(\lambda_0) - \lambda_0 \frac{dn(\lambda_0)}{d\lambda_0}}$$

where c is the free space velocity of light. Find the time taken for the electromagnetic pulse to travel a distance D.

Introduction: We are asked to derive the expression for group velocity V_g of electromagnetic waves in a dispersive medium, expressed in terms of the wavelength-dependent refractive index $n(\lambda)$, evaluated at λ_0. Then, using the derived expression, we are to find the time taken for a pulse to propagate a distance D through the medium.

Solution:

The phase velocity of electromagnetic waves in a medium is:

$$v_p = \frac{c}{n(\lambda)}$$

The angular frequency ω is related to wave number k by:

$$\omega = \frac{2\pi c}{\lambda}, \quad k = \frac{2\pi n(\lambda)}{\lambda}$$

Group velocity is defined as:

$$V_g = \frac{d\omega}{dk}$$

To compute this, we write ω and k as functions of λ:

$$\omega(\lambda) = \frac{2\pi c}{\lambda}$$
$$k(\lambda) = \frac{2\pi n(\lambda)}{\lambda}$$

Then:

$$\frac{d\omega}{d\lambda} = -\frac{2\pi c}{\lambda^2}, \quad \frac{dk}{d\lambda} = \frac{2\pi}{\lambda^2}\left[\lambda \frac{dn}{d\lambda} - n(\lambda)\right]$$

Thus:

$$\begin{aligned} V_g &= \frac{d\omega/d\lambda}{dk/d\lambda} \\ &= \frac{-\frac{2\pi c}{\lambda^2}}{\frac{2\pi}{\lambda^2}\left[\lambda \frac{dn}{d\lambda} - n(\lambda)\right]} \\ &= \frac{c}{n(\lambda) - \lambda \frac{dn}{d\lambda}} \end{aligned}$$

Evaluating this at λ_0, we get:

$$V_g = \frac{c}{n(\lambda_0) - \lambda_0 \frac{dn(\lambda_0)}{d\lambda_0}}$$

Time taken for the pulse to travel a distance D:

If the pulse propagates at group velocity V_g, then the time taken is:

$$t = \frac{D}{V_g} = \frac{D}{\dfrac{c}{n(\lambda_0) - \lambda_0 \dfrac{dn(\lambda_0)}{d\lambda_0}}} = \frac{D}{c}\left[n(\lambda_0) - \lambda_0 \frac{dn(\lambda_0)}{d\lambda_0}\right]$$

Conclusion:

The group velocity of an electromagnetic wave in a dispersive medium with wavelength-dependent refractive index $n(\lambda)$ is:

$$V_g = \frac{c}{n(\lambda_0) - \lambda_0 \frac{dn(\lambda_0)}{d\lambda_0}}$$

The time taken for a pulse to travel a distance D in such a medium is:

$$t = \frac{D}{c}\left[n(\lambda_0) - \lambda_0 \frac{dn(\lambda_0)}{d\lambda_0}\right]$$

5 The motion of a damped mechanical oscillator is represented by $m\ddot{x} + a\dot{x} + \beta x = 0$ where m, a and β are constants. The oscillator is critically damped. The system is given an impulse at $x = 0$ and $t = 0$, resulting in an initial velocity v. After how much time the system experiences maximum displacement?

Introduction: This problem involves a critically damped harmonic oscillator described by the differential equation:

$$m\ddot{x} + a\dot{x} + \beta x = 0$$

We are told that the system is critically damped and initially at rest position $x(0) = 0$ but is given an initial velocity v at $t = 0$. The objective is to determine the time at which the displacement $x(t)$ reaches its maximum value.

Solution:

For critical damping, the damping coefficient satisfies:

$$\frac{a^2}{4m^2} = \frac{\beta}{m} \Rightarrow \text{Characteristic equation has a repeated root}$$

The general solution for a critically damped system is:

$$x(t) = (A + Bt)e^{-\gamma t}, \quad \text{where } \gamma = \frac{a}{2m}$$

Given initial conditions:

$$\begin{aligned} x(0) &= 0 \Rightarrow A = 0 \\ \dot{x}(0) &= v = Be^0 - \gamma(A + B \cdot 0)e^0 = B \end{aligned}$$

So the displacement becomes:

$$x(t) = vte^{-\gamma t}$$

To find the time of maximum displacement, set $\frac{dx}{dt} = 0$:

$$\frac{dx}{dt} = ve^{-\gamma t}(1 - \gamma t) = 0$$
$$\Rightarrow 1 - \gamma t = 0 \Rightarrow t = \frac{1}{\gamma} = \frac{2m}{a}$$

Conclusion:

The system reaches its maximum displacement at time:

$$t = \frac{2m}{a}$$

This result is derived using the form of the solution for a critically damped oscillator and applying the condition for extremum of the displacement function.

6 Show that a travelling wave on the string, clamped on both the ends, undergoes a phase change of π. Hence obtain the time-independent form of the wave equation representing a standing wave on the string.

Introduction: A string fixed at both ends imposes boundary conditions that lead to the formation of standing waves due to the superposition of incident and reflected travelling waves. At a rigid boundary (clamped end), a wave reflects with a phase change of π (i.e., a sign reversal). The objective is to show this phase shift and then derive the spatial (time-independent) form of the wave equation representing a standing wave.

Solution:

Consider a travelling wave on a string given by:

$$y_i(x,t) = A\sin(kx - \omega t)$$

This wave travels in the positive x-direction toward a clamped end at $x = L$.

Proof of Phase Change of :

At the clamped boundary ($x = L$), the displacement must be zero at all times:

$$y(x = L, t) = 0 \quad \text{for all } t$$

Let the reflected wave be of the form:

$$y_r(x,t) = B\sin(k(2L - x) - \omega t + \phi)$$

where B is the amplitude and ϕ is the phase change upon reflection.

The total displacement is:

$$y(x,t) = y_i(x,t) + y_r(x,t) = A\sin(kx - \omega t) + B\sin(k(2L - x) - \omega t + \phi)$$

Applying the boundary condition at $x = L$:

$$y(L,t) = A\sin(kL - \omega t) + B\sin(kL - \omega t + \phi) = 0$$

For this to be satisfied for all values of t, we need:

$$A\sin(kL - \omega t) + B\sin(kL - \omega t + \phi) = 0$$

This requires $B = A$ and $\phi = \pi$, giving us:

$$y_r(x,t) = A\sin(k(2L - x) - \omega t + \pi) = -A\sin(k(2L - x) - \omega t)$$

Since $k(2L - x) = k(L + (L - x)) = kL + k(L - x)$, and noting that the reflected wave travels in the negative x-direction, we can write:

$$y_r(x,t) = -A\sin(kx + \omega t)$$

Thus, the phase change upon reflection is π.

Standing Wave Formation:

The total displacement is:

$$\begin{aligned} y(x,t) &= y_i(x,t) + y_r(x,t) \\ &= A\sin(kx - \omega t) - A\sin(kx + \omega t) \end{aligned}$$

Using the trigonometric identity:

$$\sin a - \sin b = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$$

We get:

$$\begin{aligned} y(x,t) &= 2A\cos\left(\frac{(kx-\omega t)+(kx+\omega t)}{2}\right)\sin\left(\frac{(kx-\omega t)-(kx+\omega t)}{2}\right) \\ &= 2A\cos(kx)\sin(-\omega t) \\ &= -2A\cos(kx)\sin(\omega t) \end{aligned}$$

This can be written as:

$$y(x,t) = [2A\cos(kx)]\sin(\omega t + \pi)$$

Boundary Conditions and Quantization:

For a string clamped at both ends ($x = 0$ and $x = L$): - At $x = 0$: $y(0,t) = 2A\cos(0)\sin(\omega t + \pi) = 0$

This suggests we need to reconsider. For both boundaries to be satisfied, we need:

$$y(x,t) = [2A\sin(kx)]\cos(\omega t)$$

This satisfies: - $y(0,t) = 2A\sin(0)\cos(\omega t) = 0$ - $y(L,t) = 2A\sin(kL)\cos(\omega t) = 0$

The second condition requires $\sin(kL) = 0$, which gives:

$$kL = n\pi \quad \Rightarrow \quad k = \frac{n\pi}{L} \quad (n = 1, 2, 3, ...)$$

Time-independent Form:

The time-independent (spatial) part of the standing wave is:

$$Y(x) = 2A\sin(kx) = 2A\sin\left(\frac{n\pi x}{L}\right)$$

Conclusion: A travelling wave reflecting off a clamped end undergoes a phase change of due to the boundary condition requirement. The superposition of incident and reflected waves forms a standing wave with time-independent spatial form:

$$Y(x) = 2A\sin\left(\frac{n\pi x}{L}\right)$$

where $n = 1, 2, 3, ...$ represents the mode number, and this form satisfies both boundary conditions $Y(0) = Y(L) = 0$.

7 During an earthquake, a horizontal shelf moves vertically. If its motion can be regarded simple harmonic, calculate the maximum value of amplitude of oscillation so that the books resting on it stay in contact with it always. Take $g = 9.8\,\mathbf{ms^{-2}}$ and $T = 0.5\,\mathbf{s}$.

Introduction: The problem describes vertical simple harmonic motion (SHM) of a shelf during an earthquake. To ensure that books on the shelf stay in contact with it at all times, we need to determine the condition under which the normal force between the books and shelf remains positive throughout the motion. We are to calculate the maximum amplitude A of this SHM under this condition, given that the period of oscillation is $T = 0.5\,\mathrm{s}$ and $g = 9.8\,\mathrm{m/s^2}$.

Solution:

Consider the forces acting on a book of mass m resting on the shelf. The forces are:

- Weight: mg (downward)
- Normal force from shelf: N (upward)

For the book to remain in contact with the shelf, we require $N \geq 0$ at all times.

Applying Newton's second law in the vertical direction (taking upward as positive):

$$N - mg = ma$$

where a is the acceleration of the shelf (and hence the book).

Therefore: $N = m(g + a)$

For contact to be maintained: $N \geq 0$, which gives us:

$$m(g + a) \geq 0$$

$$g + a \geq 0$$
$$a \geq -g$$

The most critical condition occurs when the shelf has maximum downward acceleration. In SHM, the maximum acceleration is:

$$a_{\max} = \omega^2 A$$

For the shelf moving downward with maximum acceleration, $a = -\omega^2 A$. The contact condition becomes:

$$-\omega^2 A \geq -g$$
$$\omega^2 A \leq g$$

The angular frequency ω is related to the time period T by:

$$\omega = \frac{2\pi}{T}$$

Substituting $T = 0.5\,\mathrm{s}$:

$$\omega = \frac{2\pi}{0.5} = 4\pi\,\mathrm{rad/s}$$

Substituting into the contact condition:

$$(4\pi)^2 A \leq g$$

$$16\pi^2 A \leq 9.8$$

$$A \leq \frac{9.8}{16\pi^2}$$

Calculating the numerical value:

$$A \leq \frac{9.8}{16 \times (3.1416)^2} = \frac{9.8}{157.9136} \approx 0.062\,\text{m}$$

Conclusion: The maximum amplitude of oscillation that ensures the books remain in contact with the shelf is approximately $0.062\,\text{m}$ or $6.2\,\text{cm}$.

8 The dispersion relation for deep water waves is given by $\omega^2 = gk + ak^3$, where g and a are constants. Obtain expressions for phase velocity and group velocity in terms of the wavelength λ. ω and k represent the angular frequency and wave number, respectively.

Introduction: The problem provides a dispersion relation for deep water waves as $\omega^2 = gk + ak^3$, where ω is the angular frequency, k is the wave number, g is the acceleration due to gravity, and a is a constant. We are to derive expressions for phase velocity v_p and group velocity v_g in terms of the wavelength λ.

Solution:

We start with the given dispersion relation:

$$\omega^2 = gk + ak^3$$

Taking square root on both sides:

$$\omega = \sqrt{gk + ak^3}$$

Phase velocity v_p is defined as:

$$v_p = \frac{\omega}{k}$$

Substituting ω:

$$v_p = \frac{1}{k}\sqrt{gk + ak^3}$$

Factor k inside the square root:

$$v_p = \sqrt{\frac{g}{k} + ak}$$

Now, express k in terms of wavelength λ:

$$k = \frac{2\pi}{\lambda}$$

Then:

$$v_p = \sqrt{\frac{g}{\frac{2\pi}{\lambda}} + a \cdot \frac{2\pi}{\lambda}} = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi a}{\lambda}}$$

Group velocity v_g is defined as:

$$v_g = \frac{d\omega}{dk}$$

Differentiate the original dispersion relation:

$$\omega = (gk + ak^3)^{1/2}$$

Using chain rule:

$$\frac{d\omega}{dk} = \frac{1}{2}(gk + ak^3)^{-1/2}(g + 3ak^2) = \frac{g + 3ak^2}{2\sqrt{gk + ak^3}}$$

Therefore:

$$v_g = \frac{g + 3ak^2}{2\sqrt{gk + ak^3}}$$

Now convert to λ using $k = \frac{2\pi}{\lambda}$ and then Substitute into the expression:

$$v_g = \frac{g + 3a\left(\frac{4\pi^2}{\lambda^2}\right)}{2\sqrt{g\left(\frac{2\pi}{\lambda}\right) + a\left(\frac{8\pi^3}{\lambda^3}\right)}} = \frac{g + \frac{12\pi^2 a}{\lambda^2}}{2\sqrt{\frac{2\pi g}{\lambda} + \frac{8\pi^3 a}{\lambda^3}}}$$

Conclusion: The expressions for phase and group velocities in terms of the wavelength λ are:

- Phase velocity:

$$v_p = \sqrt{\frac{g\lambda}{2\pi} + \frac{2\pi a}{\lambda}}$$

- Group velocity:

$$v_g = \frac{g + \frac{12\pi^2 a}{\lambda^2}}{2\sqrt{\frac{2\pi g}{\lambda} + \frac{8\pi^3 a}{\lambda^3}}}$$

  These describe how wave packets propagate through deep water in the presence of both gravitational and capillary effects.

9 The displacement associated with a three-dimensional plane wave is given by $\Psi(x, y, z, t) = a\cos\left[\frac{\sqrt{3}}{2}kx + \frac{1}{2}ky - \omega t\right]$. Calculate the angles made by the propagating wave with the x, y and z-axes.

Introduction: The problem provides the displacement of a three-dimensional plane wave in the form of a cosine function. The spatial part of the wave argument represents the scalar product $\vec{k}\cdot\vec{r}$, which defines the direction of wave propagation. Our task is to find the angles the wave vector $\vec{k}$ makes with the x-, y-, and z-axes.

Solution:

The wave function is:

$$\Psi(x, y, z, t) = a\cos\left[\frac{\sqrt{3}}{2}kx + \frac{1}{2}ky - \omega t\right]$$

We can identify the direction of the wave vector $\vec{k}$ from the argument of the cosine function. It has components:

$$\vec{k} = \left(\frac{\sqrt{3}}{2}k, \frac{1}{2}k, 0\right)$$

This vector represents the direction of propagation. We normalize it to find the direction cosines.

Magnitude of the wave vector $\vec{k}$:

$$|\vec{k}| = \sqrt{\left(\frac{\sqrt{3}}{2}k\right)^2 + \left(\frac{1}{2}k\right)^2 + 0^2} = k\sqrt{\frac{3}{4} + \frac{1}{4}} = k$$

Let θ_x, θ_y, and θ_z be the angles made by $\vec{k}$ with the x, y, and z axes respectively.

Using direction cosines:

$$\cos\theta_x = \frac{k_x}{|\vec{k}|} = \frac{\frac{\sqrt{3}}{2}k}{k} = \frac{\sqrt{3}}{2}$$

$$\cos\theta_y = \frac{k_y}{|\vec{k}|} = \frac{\frac{1}{2}k}{k} = \frac{1}{2}$$

$$\cos\theta_z = \frac{k_z}{|\vec{k}|} = \frac{0}{k} = 0$$

Compute angles using inverse cosine:

- $\theta_x = \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = 30^\circ$
- $\theta_y = \cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$
- $\theta_z = \cos^{-1}(0) = 90^\circ$

Conclusion: The wave vector makes the following angles with the coordinate axes:

- 30° with the x-axis,

- 60° with the y-axis,
- 90° with the z-axis.

This indicates that the wave propagates in the xy-plane, making specific angles with the coordinate directions.

10 In a certain engine, a piston undergoes vertical SHM with an amplitude of 10 cm. A washer rests on the top of the piston. As the motor is slowly speeded up, at what frequency will the washer no longer stay in contact with the piston?

Introduction: This problem considers a washer placed atop a piston undergoing vertical simple harmonic motion (SHM) with amplitude $A = 10\,\text{cm} = 0.1\,\text{m}$. As the frequency increases, the acceleration of the piston increases. We need to find the critical frequency at which the washer loses contact with the piston.

Solution:

Let's analyze the forces on the washer of mass m:

- Weight: mg (downward)
- Normal force from piston: N (upward)

For the washer to remain in contact: $N \geq 0$

Applying Newton's second law (taking upward as positive):

$$N - mg = ma$$

$$N = m(g + a)$$

For contact to be maintained: $N \geq 0$, which requires:

$$g + a \geq 0$$

$$a \geq -g$$

The most critical condition occurs when the piston has maximum downward acceleration.

For vertical SHM, if we write the position as $y(t) = A\cos(\omega t)$ (taking upward as positive), then:

$$a(t) = -\omega^2 A\cos(\omega t)$$

The maximum downward acceleration is:

$$a_{\text{max, down}} = -\omega^2 A$$

At the point of losing contact:

$$-\omega^2 A = -g$$

$$\omega^2 A = g$$

Solving for angular frequency:

$$\omega^2 = \frac{g}{A} = \frac{9.8}{0.1} = 98$$

$$\omega = \sqrt{98} = 7\sqrt{2} \approx 9.899\,\text{rad/s}$$

Converting to frequency:

$$f = \frac{\omega}{2\pi} = \frac{9.899}{2\pi} \approx 1.575\,\text{Hz}$$

Conclusion: The washer will lose contact with the piston when the frequency reaches approximately 1.58 Hz. This occurs at the top of the piston's motion when the piston accelerates downward with acceleration equal to g.