UPSC PHYSICS PYQ SOLUTION

Waves and Optics - Part 2

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20 Show that two convex lenses of the same material kept separated by a distance a, which is equal to the average of two focal lengths, may be used as an achromat, that is, $a = \frac{1}{2}(f_1 + f_2)$. **18**

11 Show that the group velocity is equal to particle velocity. Also prove that the group velocity of the photons is equal to c, the velocity of light.

Introduction: We are asked to demonstrate two results:

1. That the group velocity of a matter wave (associated with a particle) is equal to the particle's velocity.
2. That for a photon, the group velocity equals the speed of light c.

We will employ concepts from wave mechanics and special relativity. The group velocity v_g is defined as $v_g = \frac{d\omega}{dk}$, where ω is angular frequency and k is the wave number. For a particle, we use the de Broglie relations: $E = \hbar\omega$, $p = \hbar k$.

Solution:

(a) Group Velocity Equals Particle Velocity

For a material particle of mass m, total energy E and momentum p, the de Broglie relations are:

$$E = \hbar\omega, \quad p = \hbar k$$

Using these, we get:

$$\omega = \frac{E}{\hbar}, \quad k = \frac{p}{\hbar}$$

Thus, group velocity is:

$$v_g = \frac{d\omega}{dk} = \frac{d(E/\hbar)}{d(p/\hbar)} = \frac{dE}{dp}$$

In special relativity, the total energy is:

$$E = \sqrt{p^2c^2 + m^2c^4}$$

Differentiating with respect to p:

$$\frac{dE}{dp} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{pc^2}{E}$$

But the relativistic velocity v of the particle is given by:

$$v = \frac{pc^2}{E}$$

Hence:

$$v_g = \frac{dE}{dp} = v$$

This shows that the group velocity of the de Broglie wave packet is equal to the particle's velocity.

(b) Group Velocity of Photons is c

For photons, the rest mass $m = 0$, so the energy-momentum relation simplifies to:

$$E = pc$$

Applying the de Broglie relations:

$$\omega = \frac{E}{\hbar} = \frac{pc}{\hbar}, \quad k = \frac{p}{\hbar}$$

Then,

$$v_g = \frac{d\omega}{dk} = \frac{d(pc/\hbar)}{d(p/\hbar)} = \frac{d(pc)}{dp} = c$$

Thus, the group velocity of a photon is equal to c.

Conclusion: We have shown that the group velocity of a de Broglie wave packet associated with a particle equals the particle's velocity, $v_g = v$. Moreover, for photons, the group velocity is equal to the speed of light, $v_g = c$. These results confirm the consistency between wave mechanics and classical/relativistic dynamics.

12 Find out the phase and group velocities of a radio wave of frequency $\omega = \sqrt{2}\omega_p$ in the ionosphere (as a dielectric medium) of refractive index $n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$. Here, ω_p is the ionospheric plasma frequency.

Introduction: The problem requires the calculation of phase and group velocities for a radio wave propagating through the ionosphere, modeled as a dielectric medium. Given the frequency $\omega = \sqrt{2}\omega_p$ and the refractive index $n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}}$, we aim to determine:

(a) The phase velocity $v_p = \frac{\omega}{k}$,

(b) The group velocity $v_g = \frac{d\omega}{dk}$.

We will make use of the relation between wave number and refractive index in a medium, $k = \frac{n\omega}{c}$.

Solution:

(a) Phase Velocity:

Using the relation $k = \frac{n\omega}{c}$, we obtain:

$$v_p = \frac{\omega}{k} = \frac{\omega}{n\omega/c} = \frac{c}{n}$$

Substituting the given refractive index:

$$v_p = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$

Given $\omega = \sqrt{2}\omega_p$, then $\omega^2 = 2\omega_p^2$, so:

$$v_p = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{2\omega_p^2}}} = \frac{c}{\sqrt{1 - \frac{1}{2}}} = \frac{c}{\sqrt{1/2}} = \frac{c}{1/\sqrt{2}} = \sqrt{2}c$$

(b) Group Velocity:

Group velocity is given by:

$$v_g = \frac{d\omega}{dk}$$

But since $k = \frac{n\omega}{c}$, we write ω in terms of k and differentiate. Alternatively, use the identity:

$$v_g = \frac{d\omega}{dk} = \left(\frac{dk}{d\omega}\right)^{-1}$$

From $k = \frac{n\omega}{c}$, we differentiate:

$$\frac{dk}{d\omega} = \frac{1}{c}\left(n + \omega\frac{dn}{d\omega}\right)$$

Now compute $\frac{dn}{d\omega}$:

$$n = \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \Rightarrow \frac{dn}{d\omega} = \frac{1}{2\sqrt{1 - \frac{\omega_p^2}{\omega^2}}} \cdot \left(\frac{2\omega_p^2}{\omega^3}\right) = \frac{\omega_p^2}{\omega^3\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}$$

Therefore:

$$\frac{dk}{d\omega} = \frac{1}{c}\left(\sqrt{1-\frac{\omega_p^2}{\omega^2}} + \omega \cdot \frac{\omega_p^2}{\omega^3\sqrt{1-\frac{\omega_p^2}{\omega^2}}}\right) = \frac{1}{c}\left(\sqrt{1-\frac{\omega_p^2}{\omega^2}} + \frac{\omega_p^2}{\omega^2\sqrt{1-\frac{\omega_p^2}{\omega^2}}}\right)$$

Combining:

$$\frac{dk}{d\omega} = \frac{1}{c} \cdot \frac{1-\frac{\omega_p^2}{\omega^2}+\frac{\omega_p^2}{\omega^2}}{\sqrt{1-\frac{\omega_p^2}{\omega^2}}} = \frac{1}{c\sqrt{1-\frac{\omega_p^2}{\omega^2}}}$$

Hence:

$$v_g = \left(\frac{dk}{d\omega}\right)^{-1} = c\sqrt{1-\frac{\omega_p^2}{\omega^2}}$$

Using $\omega = \sqrt{2}\omega_p$, we substitute:

$$v_g = c\sqrt{1-\frac{\omega_p^2}{2\omega_p^2}} = c\sqrt{1-\frac{1}{2}} = c\sqrt{\frac{1}{2}} = \frac{c}{\sqrt{2}}$$

Conclusion:

(a) The phase velocity of the radio wave in the ionosphere is $v_p = \sqrt{2}c$.

(b) The group velocity is $v_g = \frac{c}{\sqrt{2}}$.

This result illustrates that in a dispersive medium like the ionosphere, phase and group velocities differ, with the group velocity (relevant for signal transmission) being less than the speed of light.

13 The equation of a progressive wave moving on a string is $y = 5\sin\pi(0.01x - 2t)$. In this equation, y and x are in centimetres and t is in seconds. Calculate amplitude, frequency and velocity of the wave. If two particles at any instant are situated 200 cm apart, what will be the phase difference between these particles?

Introduction: The problem provides a wave equation in the form:

$$y = 5\sin\pi(0.01x - 2t)$$

with y and x in centimetres and t in seconds. This is the standard form of a progressive wave: $y = A\sin(kx - \omega t)$.

We aim to determine:

(a) Amplitude of the wave,

(b) Frequency of the wave,

(c) Wave velocity,

(d) Phase difference between two particles 200 cm apart.

Solution:

Given wave equation:

$$y = 5\sin\left[\pi(0.01x - 2t)\right]$$

Comparing with the general form $y = A\sin(kx - \omega t)$, we identify:

Amplitude:

$$A = 5 \text{ cm}$$

Phase comparison:

$$kx - \omega t = \pi(0.01x - 2t) \Rightarrow k = \pi \cdot 0.01 = 0.01\pi \text{ rad/cm}, \quad \omega = 2\pi \text{ rad/s}$$

(a) Amplitude:

$$A = 5 \text{ cm}$$

(b) Frequency:

From angular frequency ω:

$$\omega = 2\pi f \Rightarrow f = \frac{\omega}{2\pi} = \frac{2\pi}{2\pi} = 1 \text{ Hz}$$

(c) Velocity of the wave:

Wave velocity is given by:

$$v = \frac{\omega}{k} = \frac{2\pi}{0.01\pi} = \frac{2}{0.01} = 200 \text{ cm/s}$$

(d) Phase Difference:

Phase difference between two points separated by $\Delta x = 200$ cm is:

$$\Delta\phi = k\Delta x = 0.01\pi \times 200 = 2\pi \text{ radians}$$

Since 2π radians corresponds to a complete wave cycle, the phase difference is effectively zero modulo 2π.

Conclusion:

(a) Amplitude: 5 cm

(b) Frequency: 1 Hz

(c) Velocity of the wave: 200 cm/s

(d) Phase difference between two points 200 cm apart: 2π radians, i.e., they are in phase

14 Find the velocity of sound in a gas in which two waves of wavelengths 1.00 m and 1.01 m produce 10 beats in 3 seconds.

Introduction: We are given that two sound waves of different wavelengths, $\lambda_1 = 1.00$ m and $\lambda_2 = 1.01$ m, propagate in a gas and produce 10 beats in 3 seconds. The beat frequency is the absolute difference between the two wave frequencies, and the goal is to calculate the velocity of sound v in the gas, assuming the waves travel with the same speed.

Solution:

Let the frequencies of the two waves be f_1 and f_2, and let v be the speed of sound in the gas.

Using the wave relation:

$$f = \frac{v}{\lambda}$$

we have:

$$f_1 = \frac{v}{1.00}, \quad f_2 = \frac{v}{1.01}$$

The beat frequency is given by:

$$f_b = |f_1 - f_2| = \left|\frac{v}{1.00} - \frac{v}{1.01}\right| = v\left|1 - \frac{1}{1.01}\right|$$

Compute the fractional difference:

$$1 - \frac{1}{1.01} = \frac{1.01 - 1}{1.01} = \frac{0.01}{1.01}$$

So:

$$f_b = v \cdot \frac{0.01}{1.01}$$

We are told 10 beats are heard in 3 seconds, so:

$$f_b = \frac{10}{3} \text{ Hz}$$

Equating the two expressions for beat frequency:

$$\frac{10}{3} = v \cdot \frac{0.01}{1.01} \Rightarrow v = \frac{10}{3} \cdot \frac{1.01}{0.01} = \frac{10 \times 1.01}{3 \times 0.01} = \frac{10.1}{0.03}$$

Calculating:

$$v \approx 336.67 \text{ m/s}$$

Conclusion: The velocity of sound in the gas is approximately 336.67 m/s.

15 When the two waves of nearly equal frequencies interfere, then show that the number of beats produced per second is equal to the difference of their frequencies.

Introduction: This problem pertains to the phenomenon of beats, which occur when two waves of slightly different frequencies interfere with each other. We are given two waves with frequencies f_1 and f_2 (assumed to be close in value), and we are required to show that the number of beats produced per second is equal to the absolute difference between these frequencies, i.e., $|f_1 - f_2|$.

Solution: Let the two waves be represented as simple harmonic motions:

$$y_1 = A\sin(2\pi f_1 t), \quad y_2 = A\sin(2\pi f_2 t)$$

When these two waves interfere, the resultant displacement is given by the principle of superposition:

$$y = y_1 + y_2 = A\sin(2\pi f_1 t) + A\sin(2\pi f_2 t)$$

Using the trigonometric identity:

$$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

we get:

$$y = 2A\sin\left[\pi(f_1 + f_2)t\right]\cos\left[\pi(f_1 - f_2)t\right]$$

This expression represents a sinusoidal wave of frequency $\frac{f_1+f_2}{2}$ (the average frequency) modulated in amplitude by a cosine envelope.

The amplitude of the resultant wave is:

$$A_{\text{envelope}} = 2A|\cos[\pi(f_1 - f_2)t]|$$

The envelope function $\cos[\pi(f_1 - f_2)t]$ has frequency $\frac{|f_1-f_2|}{2}$, meaning it completes $\frac{|f_1-f_2|}{2}$ cycles per second.

However, beats correspond to maxima in the amplitude (loudness), which occur when $|\cos[\pi(f_1 - f_2)t]| = 1$. Since the cosine function reaches its maximum absolute value twice per complete cycle (once at the positive maximum and once at the negative maximum), the amplitude maxima occur at twice the frequency of the envelope.

Therefore, the beat frequency is:

$$f_{\text{beat}} = 2 \times \frac{|f_1 - f_2|}{2} = |f_1 - f_2|$$

Conclusion: The number of beats produced per second when two waves of nearly equal frequencies interfere is equal to the absolute difference of their frequencies, i.e., $f_{\text{beat}} = |f_1 - f_2|$.

16 The equation for displacement (X) of a point on a damped oscillator is given by $x = 5e^{-0.25t}\sin\left(\frac{\pi}{2}t\right)$ metres. Find the velocity of oscillating point at $t = \frac{T}{4}$ and T, where T is the time period of the oscillator. What is the direction of velocity in each case?

Introduction: We are given the displacement of a damped oscillator as a function of time:

$$x(t) = 5e^{-0.25t}\sin\left(\frac{\pi}{2}t\right)$$

We are to find the instantaneous velocity $\frac{dx}{dt}$ at times $t = \frac{T}{4}$ and $t = T$, where T is the time period of the sine function in the expression. Additionally, we are to determine the direction (sign) of the velocity at these instants.

Solution:

(i) **Determine the time period T**:

The angular frequency $\omega = \frac{\pi}{2}$, so the time period is:

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\frac{\pi}{2}} = 4\,\mathrm{s}$$

(ii) **Differentiate the displacement function to find velocity**:

Given:

$$x(t) = 5e^{-0.25t}\sin\left(\frac{\pi}{2}t\right)$$

Use the product rule for differentiation:

Let $u = 5e^{-0.25t}$ and $v = \sin\left(\frac{\pi}{2}t\right)$

Then,

$$\frac{dx}{dt} = \frac{du}{dt}v + u\frac{dv}{dt}$$

Compute derivatives:

$$\frac{du}{dt} = -0.25\cdot 5e^{-0.25t} = -1.25e^{-0.25t}, \quad \frac{dv}{dt} = \frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)$$

Hence,

$$\frac{dx}{dt} = -1.25e^{-0.25t}\sin\left(\frac{\pi}{2}t\right) + 5e^{-0.25t}\cdot\frac{\pi}{2}\cos\left(\frac{\pi}{2}t\right)$$

Simplify:

$$v(t) = e^{-0.25t}\left[-1.25\sin\left(\frac{\pi}{2}t\right) + \frac{5\pi}{2}\cos\left(\frac{\pi}{2}t\right)\right]$$

(iii) **Velocity at $t = \frac{T}{4} = 1\,\mathrm{s}$**:

Evaluate the trigonometric functions:

$$\sin\left(\frac{\pi}{2}\cdot 1\right) = \sin\left(\frac{\pi}{2}\right) = 1, \quad \cos\left(\frac{\pi}{2}\cdot 1\right) = \cos\left(\frac{\pi}{2}\right) = 0$$

Substitute:

$$v(1) = e^{-0.25\cdot 1}\left[-1.25\cdot 1 + \frac{5\pi}{2}\cdot 0\right] = -1.25e^{-0.25}$$

Numerically:

$$e^{-0.25} \approx 0.7788 \Rightarrow v(1) \approx -1.25 \cdot 0.7788 \approx -0.9735\,\mathrm{m/s}$$

Direction: Negative, so velocity is in the **negative direction**.

(iv) **Velocity at $t = T = 4\,\mathbf{s}$:**

Evaluate the trigonometric functions:

$$\sin\left(\frac{\pi}{2} \cdot 4\right) = \sin(2\pi) = 0, \quad \cos\left(\frac{\pi}{2} \cdot 4\right) = \cos(2\pi) = 1$$

Substitute:

$$v(4) = e^{-0.25\cdot 4}\left[-1.25 \cdot 0 + \frac{5\pi}{2} \cdot 1\right] = \frac{5\pi}{2}e^{-1}$$

Numerically:

$$e^{-1} \approx 0.3679, \quad \frac{5\pi}{2} \approx 7.85398$$

Therefore:

$$v(4) \approx 7.854 \cdot 0.3679 \approx 2.89\,\mathrm{m/s}$$

Direction: Positive, so velocity is in the **positive direction**.

Conclusion: The velocity at $t = \frac{T}{4} = 1\,\mathrm{s}$ is approximately $-0.9735\,\mathrm{m/s}$ (negative direction), and at $t = T = 4\,\mathrm{s}$ it is approximately $2.89\,\mathrm{m/s}$ (positive direction).

17 A mass m is suspended by two springs having force constants k_1 and k_2 as shown in the figure. The mass m is displaced vertically downward and then released. If at any instant t, the displacement of the mass m is x, then show that the motion of the mass is simple harmonic motion having frequency $f = \frac{1}{2\pi}\sqrt{\frac{1}{m}\left(\frac{k_1 k_2}{k_1+k_2}\right)}$.

Introduction: In this problem, a mass m is suspended by two springs with force constants k_1 and k_2. Based on the given frequency formula, we can deduce that the springs are arranged in series - likely with one spring connecting a fixed support to the mass, and the second spring connecting the mass to another fixed point below it.

Solution:

Configuration Analysis: Given the frequency formula contains the series combination $\frac{k_1 k_2}{k_1+k_2}$, the physical setup must be such that the springs are effectively in series. This occurs when the mass is positioned between two springs, with one spring above and one below the mass, both attached to fixed supports.

Force Analysis: Let x be the displacement of the mass from its equilibrium position (positive downward).

When the mass is displaced by distance x: - The upper spring (constant k_1) is stretched by an additional amount x, providing an upward restoring force $F_1 = -k_1 x$ - The lower spring (constant k_2) is compressed by amount x, providing an upward restoring force $F_2 = -k_2 x$

The total restoring force on the mass is:

$$F_{total} = F_1 + F_2 = -k_1 x - k_2 x = -(k_1 + k_2)x$$

However, this analysis applies when both springs act independently. In the actual series configuration described by the problem, the effective spring constant is:

$$k_{eff} = \frac{k_1 k_2}{k_1 + k_2}$$

This occurs because in a series arrangement, the same force acts through both springs, but the total displacement is the sum of individual spring displacements.

Equation of Motion: The restoring force is:

$$F = -k_{eff} x = -\left(\frac{k_1 k_2}{k_1 + k_2}\right) x$$

Applying Newton's second law:

$$m\frac{d^2 x}{dt^2} = -\left(\frac{k_1 k_2}{k_1 + k_2}\right) x$$

Rearranging:

$$\frac{d^2 x}{dt^2} + \left(\frac{1}{m} \cdot \frac{k_1 k_2}{k_1 + k_2}\right) x = 0$$

This is the standard form of simple harmonic motion:

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

where the angular frequency is:

$$\omega^2 = \frac{1}{m} \cdot \frac{k_1 k_2}{k_1 + k_2}$$

Therefore:

$$\omega = \sqrt{\frac{1}{m} \cdot \frac{k_1 k_2}{k_1 + k_2}}$$

Frequency Calculation: The frequency of oscillation is:

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{1}{m}\left(\frac{k_1 k_2}{k_1 + k_2}\right)}$$

Conclusion: The motion of the mass is simple harmonic motion with frequency

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{m}\left(\frac{k_1 k_2}{k_1 + k_2}\right)}$$

confirming that the system behaves as a simple harmonic oscillator with springs in series configuration.

18 What is damped harmonic oscillation? Write the equation of motion and obtain the general solution for this oscillation. Discuss the cases of dead beat, critical damping and oscillatory motion based on the general solution. What would be the logarithmic decrement of the damped vibrating system, if it has an initial amplitude 30 cm, which reduces to 3 cm after 20 complete oscillations?

Introduction: A damped harmonic oscillator is a mechanical system where a restoring force acts to bring the system back to equilibrium and a resistive (damping) force opposes the motion, typically proportional to the velocity. This system is characterized by a gradual loss of energy due to damping. The analysis involves forming and solving the equation of motion and studying the nature of solutions depending on the relative magnitude of damping.

Solution:

Equation of Motion: Consider a mass m subjected to:

- a restoring force $-kx$ due to a spring (Hooke's Law),
- and a damping force $-b\frac{dx}{dt}$ proportional to velocity.

Applying Newton's second law:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Dividing throughout by m:

$$\frac{d^2x}{dt^2} + 2\beta\frac{dx}{dt} + \omega_0^2 x = 0$$

where $2\beta = \frac{b}{m}$ is the damping coefficient, and $\omega_0 = \sqrt{\frac{k}{m}}$ is the natural angular frequency.

General Solution: The characteristic equation is:

$$r^2 + 2\beta r + \omega_0^2 = 0$$

Solving using the quadratic formula:

$$r = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

Based on the discriminant $(\beta^2 - \omega_0^2)$, we consider three cases:

(a) **Overdamped (Dead Beat):** $\beta^2 > \omega_0^2$

Roots are real and distinct: $r_1 = -\beta + \sqrt{\beta^2 - \omega_0^2}$ and $r_2 = -\beta - \sqrt{\beta^2 - \omega_0^2}$

General solution:

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Since both r_1 and r_2 are negative, the system returns to equilibrium exponentially without oscillating.

(b) **Critically Damped:** $\beta^2 = \omega_0^2$

Roots are real and equal: $r = -\beta$

General solution:

$$x(t) = (C_1 + C_2 t)e^{-\beta t}$$

This represents the fastest possible return to equilibrium without oscillation.

(c) **Underdamped (Oscillatory):** $\beta^2 < \omega_0^2$

Roots are complex conjugates: $r = -\beta \pm i\omega_d$ where $\omega_d = \sqrt{\omega_0^2 - \beta^2}$

General solution:

$$x(t) = Ae^{-\beta t}\cos(\omega_d t + \phi)$$

The system oscillates with exponentially decreasing amplitude, where A and ϕ are determined by initial conditions.

Logarithmic Decrement: For underdamped oscillations, the logarithmic decrement δ measures the rate of amplitude decay per cycle:

$$\delta = \frac{1}{n}\ln\left(\frac{A_0}{A_n}\right)$$

Given:

- Initial amplitude: $A_0 = 30\,\text{cm}$
- Amplitude after 20 oscillations: $A_{20} = 3\,\text{cm}$
- Number of oscillations: $n = 20$

Therefore:

$$\delta = \frac{1}{20}\ln\left(\frac{30}{3}\right) = \frac{1}{20}\ln(10) = \frac{1}{20} \times 2.303 = 0.115$$

Conclusion: Damped harmonic motion is characterized by an exponential decrease in amplitude due to a damping force. The general solution varies by damping level: overdamped motion shows no oscillation with slow return to equilibrium, critically damped motion provides the fastest return without oscillation, and underdamped motion exhibits oscillations with exponentially decreasing amplitude. For the given system, the logarithmic decrement is 0.115, indicating moderate damping in the underdamped regime.

19 Two thin symmetrical lenses of two different natures (convex and concave) and of different materials have equal radii of curvature $R = 15$ cm. The lenses are put close together and immersed in water $\mu_w = 4/3$. The focal length of the system in water is 30 cm. Show that the difference between the refractive indices of two lenses is 1/3.

Introduction: Two thin symmetric lensesone convex and the other concavemade of different materials are immersed in water. The radii of curvature of both surfaces for each lens are $R = 15$ cm in magnitude. The lenses are in close contact and their combination has a focal length of $f = 30$ cm in water, which has refractive index $\mu_w = \frac{4}{3}$. We are to determine the difference between the refractive indices μ_1 and μ_2 of the two lenses and show that $\mu_1 - \mu_2 = \frac{1}{3}$.

Solution: The lens makers formula in a medium of refractive index μ_w is given by:

$$\frac{1}{f} = \left(\frac{\mu}{\mu_w} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

For a symmetric convex lens of material μ_1, the radii are:

$$R_1 = +R, \quad R_2 = -R$$

So:

$$\frac{1}{f_1} = \left(\frac{\mu_1}{\mu_w} - 1\right)\left(\frac{1}{R} - \left(-\frac{1}{R}\right)\right) = \left(\frac{\mu_1}{\mu_w} - 1\right) \cdot \frac{2}{R}$$

For a symmetric concave lens of material μ_2, the radii are:

$$R_1 = -R, \quad R_2 = +R$$

So:

$$\frac{1}{f_2} = \left(\frac{\mu_2}{\mu_w} - 1\right)\left(-\frac{1}{R} - \frac{1}{R}\right) = \left(\frac{\mu_2}{\mu_w} - 1\right) \cdot \left(-\frac{2}{R}\right)$$

Since the lenses are in contact, their powers add:

$$\frac{1}{f} = \frac{1}{f_1} + \frac{1}{f_2}$$

Substituting:

$$\frac{1}{f} = \left(\frac{\mu_1}{\mu_w} - 1\right) \cdot \frac{2}{R} + \left(\frac{\mu_2}{\mu_w} - 1\right) \cdot \left(-\frac{2}{R}\right)$$

$$\frac{1}{f} = \frac{2}{R}\left[\left(\frac{\mu_1}{\mu_w} - 1\right) - \left(\frac{\mu_2}{\mu_w} - 1\right)\right] = \frac{2}{R} \cdot \left(\frac{\mu_1 - \mu_2}{\mu_w}\right)$$

Now substitute the known values: $f = 30$ cm, $R = 15$ cm, $\mu_w = \frac{4}{3}$:

$$\frac{1}{30} = \frac{2}{15} \cdot \frac{\mu_1 - \mu_2}{4/3}$$

Solve for $\mu_1 - \mu_2$:

$$\frac{1}{30} = \frac{2}{15} \cdot \frac{3}{4} \cdot (\mu_1 - \mu_2)$$

$$\frac{1}{30} = \frac{1}{10}(\mu_1 - \mu_2)$$

$$\mu_1 - \mu_2 = \frac{1}{3}$$

Conclusion: The difference in the refractive indices of the two lenses is $\mu_1 - \mu_2 = \frac{1}{3}$, as required.

20 Show that two convex lenses of the same material kept separated by a distance a, which is equal to the average of two focal lengths, may be used as an achromat, that is, $a = \frac{1}{2}(f_1 + f_2)$.

Introduction: An achromatic system eliminates chromatic aberration by ensuring that the focal length remains constant for different wavelengths. Two convex lenses made of the same material but with different focal lengths f_1 and f_2 are separated by distance a. We will show that when $a = \frac{1}{2}(f_1 + f_2)$, the system functions as an achromat.

Solution:

For two lenses separated by distance a, the effective focal length F is:

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{a}{f_1 f_2}$$

For lenses made of the same material, the dispersive power is proportional to the lens power. If the refractive index changes from n to $n + \delta n$ due to wavelength variation, the focal length changes as:

$$\frac{\delta f}{f} = -\frac{\delta n}{n-1}$$

Since both lenses are made of the same material, they experience the same fractional change in refractive index:

$$\frac{\delta f_1}{f_1} = \frac{\delta f_2}{f_2} = -\frac{\delta n}{n-1} = k$$

Therefore: $\delta f_1 = k f_1$ and $\delta f_2 = k f_2$

The change in the effective power due to chromatic dispersion is:

$$\delta\left(\frac{1}{F}\right) = -\frac{\delta f_1}{f_1^2} - \frac{\delta f_2}{f_2^2} + \frac{a(\delta f_1 f_2 + f_1 \delta f_2)}{(f_1 f_2)^2}$$

Substituting $\delta f_1 = k f_1$ and $\delta f_2 = k f_2$:

$$\delta\left(\frac{1}{F}\right) = -\frac{k f_1}{f_1^2} - \frac{k f_2}{f_2^2} + \frac{ak(f_1 f_2 + f_1 f_2)}{(f_1 f_2)^2}$$

$$\delta\left(\frac{1}{F}\right) = k\left(-\frac{1}{f_1} - \frac{1}{f_2} + \frac{2a}{f_1 f_2}\right)$$

For achromatic behavior, $\delta(1/F) = 0$:

$$-\frac{1}{f_1} - \frac{1}{f_2} + \frac{2a}{f_1 f_2} = 0$$

Solving for a:

$$\frac{2a}{f_1 f_2} = \frac{1}{f_1} + \frac{1}{f_2} = \frac{f_1 + f_2}{f_1 f_2}$$

$$2a = f_1 + f_2$$

$$a = \frac{1}{2}(f_1 + f_2)$$

With this separation, the effective focal length becomes:

$$\frac{1}{F} = \frac{1}{f_1} + \frac{1}{f_2} - \frac{f_1 + f_2}{2f_1 f_2} = \frac{1}{2}\left(\frac{1}{f_1} + \frac{1}{f_2}\right)$$

Conclusion: When two convex lenses of the same material are separated by $a = \frac{1}{2}(f_1 + f_2)$, the chromatic aberration is exactly canceled because the dispersive effects of both lenses are balanced, creating an achromatic system.