UPSC PHYSICS PYQ SOLUTION
Waves and Optics - Part 3

Contents

31 Obtain the system matrix for a thick lens and derive the thin lens formula.

Introduction: A thick lens is analyzed using ray transfer matrices (ABCD matrices) in geometric optics. The system consists of three sequential operations: refraction at the first surface, propagation through the lens material, and refraction at the second surface. We use the paraxial approximation and establish sign conventions where distances are positive to the right of surfaces and radii are positive if centers of curvature lie to the right.

Given Parameters:

- n: refractive index of the lens material
- R_1, R_2: radii of curvature of first and second surfaces
- d: thickness of the lens along the optical axis
- Medium: air (refractive index = 1) on both sides

Solution:

Individual Matrix Elements

(i) Refraction at first surface (air to lens): The refraction matrix for a spherical surface is:

$$M_1 = \begin{bmatrix} 1 & 0 \\ -\frac{n_2-n_1}{n_2R_1} & \frac{n_1}{n_2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{n-1}{nR_1} & \frac{1}{n} \end{bmatrix}$$

(ii) Translation through lens material:

$$M_2 = \begin{bmatrix} 1 & \frac{d}{n} \\ 0 & 1 \end{bmatrix}$$

(iii) Refraction at second surface (lens to air):

$$M_3 = \begin{bmatrix} 1 & 0 \\ -\frac{1-n}{R_2} & n \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{bmatrix}$$

System Matrix for Thick Lens

The overall system matrix is:

$$M = M_3 \cdot M_2 \cdot M_1$$

First, calculate $M_2 \cdot M_1$:

$$M_2M_1 = \begin{bmatrix} 1 & \frac{d}{n} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{n-1}{nR_1} & \frac{1}{n} \end{bmatrix} = \begin{bmatrix} 1 - \frac{d(n-1)}{n^2R_1} & \frac{d}{n^2} \\ -\frac{n-1}{nR_1} & \frac{1}{n} \end{bmatrix}$$

Now multiply by M_3:

$$M = M_3 \cdot (M_2M_1) = \begin{bmatrix} 1 & 0 \\ \frac{n-1}{R_2} & n \end{bmatrix} \begin{bmatrix} 1 - \frac{d(n-1)}{n^2R_1} & \frac{d}{n^2} \\ -\frac{n-1}{nR_1} & \frac{1}{n} \end{bmatrix}$$

Performing the matrix multiplication:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where:

$$A = 1 - \frac{d(n-1)}{n^2 R_1}$$

$$B = \frac{d}{n^2}$$

$$C = \frac{n-1}{R_2}\left(1 - \frac{d(n-1)}{n^2 R_1}\right) - n \cdot \frac{n-1}{nR_1}$$

$$C = (n-1)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) - \frac{d(n-1)^2}{n^2 R_1 R_2}$$

$$D = \frac{d(n-1)}{nR_2} + 1$$

Thin Lens Limit

For a thin lens, $d \to 0$, so the system matrix becomes:

$$M_{\text{thin}} = \begin{bmatrix} 1 & 0 \\ (n-1)\left(\frac{1}{R_2} - \frac{1}{R_1}\right) & 1 \end{bmatrix}$$

Derivation of Thin Lens Formula

For a thin lens, the ABCD matrix relates object and image positions. If an object is at distance s from the lens and the image forms at distance s', then:

$$\begin{bmatrix} y' \\ \theta' \end{bmatrix} = \begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ \theta \end{bmatrix}$$

where $\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$ from the lens-maker's formula.

For an object ray parallel to the axis ($\theta = 0$), the combined matrix gives:

$$\begin{bmatrix} 1 & s' \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{bmatrix} \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{s'}{f} & s + s' - \frac{ss'}{f} \\ -\frac{1}{f} & 1 - \frac{s}{f} \end{bmatrix}$$

For the ray to converge to a point (image), the (1,2) element must be zero:

$$s + s' - \frac{ss'}{f} = 0$$

Dividing by ss':

$$\frac{1}{s'} + \frac{1}{s} = \frac{1}{f}$$

This is the thin lens formula.

Conclusion: The thick lens system matrix accounts for finite thickness through three matrix operations. In the thin lens limit, we obtain both the lens-maker's formula:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

and the thin lens formula:

$$\frac{1}{f} = \frac{1}{s} + \frac{1}{s'}$$

where s is the object distance, s' is the image distance, and f is the focal length.

32 An optical beam of spectral width 7.5 GHz at wavelength $\lambda = 600$ nm is incident normally on Fabry-Perot etalon of thickness 100 mm. Taking refractive index unity, find the number of axial modes which can be supported by the etalon.

Introduction: We are given a Fabry-Perot etalon of thickness $L = 100\,\text{mm} = 0.1\,\text{m}$ and refractive index $n = 1$. An optical beam of central wavelength $\lambda = 600\,\text{nm}$ and spectral width $\Delta\nu = 7.5\,\text{GHz}$ is incident normally. The problem asks to determine the number of axial (longitudinal) modes that can be supported by the etalon within this spectral width. Axial modes in a Fabry-Perot cavity are separated by the free spectral range (FSR), which is determined by the cavity geometry and refractive index.

Solution:

The free spectral range (FSR) of a Fabry-Perot etalon is given by:

$$\text{FSR} = \frac{c}{2nL}$$

where:

- $c = 3 \times 10^8\,\text{m/s}$ (speed of light),
- $n = 1$ (refractive index),
- $L = 0.1\,\text{m}$ (thickness of the etalon).

Substituting values:

$$\text{FSR} = \frac{3 \times 10^8}{2 \times 1 \times 0.1} = \frac{3 \times 10^8}{0.2} = 1.5 \times 10^9\,\text{Hz} = 1.5\,\text{GHz}$$

The number of axial modes N that can be supported within the spectral width $\Delta\nu = 7.5\,\text{GHz}$ is:

$$N = \frac{\Delta\nu}{\text{FSR}} = \frac{7.5\,\text{GHz}}{1.5\,\text{GHz}} = 5$$

Conclusion: The Fabry-Perot etalon of thickness $100\,\text{mm}$ and refractive index 1 can support **5 axial modes** for an incident beam of spectral width $7.5\,\text{GHz}$ centered at $\lambda = 600\,\text{nm}$.

33 Describe Michelson interferometer for evaluation of coherence length of an optical beam. Calculate coherence length of a light beam of wavelength 600 nm with spectral width of 0.01 nm.

Introduction: The Michelson interferometer is an optical instrument used to measure interference patterns by splitting and recombining a beam of light. It is particularly useful for evaluating the coherence properties of light, including the coherence length, which characterizes the maximum path difference over which interference fringes remain visible. The coherence length is inversely related to the spectral width of the source. We are given a light beam with central wavelength $\lambda = 600\,\mathrm{nm}$ and spectral width $\Delta\lambda = 0.01\,\mathrm{nm}$, and we are to compute its coherence length.

Solution:

Michelson Interferometer Description: The Michelson interferometer consists of:

- A beam splitter that divides the incoming light into two perpendicular beams
- Two mirrors: one fixed (M1) and one movable (M2) that reflect the beams back
- A detector where the beams recombine to produce interference fringes

Coherence Length Evaluation Procedure:

- The optical path difference (OPD) between the two arms is: OPD = 2(d_2 - d_1), where d_1 and d_2 are the distances from beam splitter to mirrors M1 and M2 respectively
- As the movable mirror M2 is displaced, the OPD changes, causing the interference pattern to shift
- The fringe visibility (contrast) is given by: V = (I_max - I_min)/(I_max + I_min)
- For a source with finite spectral width, different wavelength components within the spectrum acquire different phase differences as OPD increases, leading to progressive loss of fringe visibility
- The coherence length L_c is defined as the OPD at which fringe visibility drops to zero (or practically becomes unobservable)
- Experimentally, by gradually increasing the mirror displacement and monitoring fringe visibility, one can determine the coherence length when fringes disappear
- This occurs because waves with different wavelengths within the spectral width $\Delta\lambda$ go in and out of phase as the path difference increases

Coherence Length Calculation: The coherence length for a source with spectral width $\Delta\lambda$ is:

$$L_c = \frac{\lambda^2}{\Delta\lambda}$$

This formula arises from the condition that phase differences between extreme wavelengths in the spectrum become 2π when the path difference equals the coherence length.

Given:

- $\lambda = 600\,\text{nm} = 600 \times 10^{-9}\,\text{m}$
- $\Delta\lambda = 0.01\,\text{nm} = 0.01 \times 10^{-9}\,\text{m}$

Substitute into the formula:

$$L_c = \frac{(600 \times 10^{-9})^2}{0.01 \times 10^{-9}} = \frac{360 \times 10^{-18}}{0.01 \times 10^{-9}} = \frac{360 \times 10^{-18}}{10^{-11}} = 36 \times 10^{-6}\,\text{m} = 36\,\mu\text{m}$$

Conclusion: The Michelson interferometer enables measurement of coherence length by observing how fringe visibility decreases with increasing optical path difference due to the finite spectral width of the source. The coherence length corresponds to the maximum OPD at which interference fringes remain observable. For a light beam of wavelength 600 nm and spectral width 0.01 nm, the coherence length is **36 micrometers**.

34 Show that two light beams polarized in perpendicular directions will not interfere.

Introduction: Interference of light arises from the superposition of coherent electric fields. For two beams to produce observable interference fringes, their electric field vectors must have a non-zero projection onto each other. This requires that they be at least partially polarized in the same direction. We aim to demonstrate that if two beams are polarized in mutually perpendicular directions, their interference term vanishes and no observable fringes are formed.

Solution:

Let the electric field vectors of the two beams be:

$$\vec{E}_1 = E_0 \cos(kx - \omega t)\hat{x}, \quad \vec{E}_2 = E_0 \cos(kx - \omega t + \phi)\hat{y}$$

where:

- $\hat{x}$ and $\hat{y}$ are unit vectors in mutually perpendicular directions,
- ϕ is the phase difference between the two waves.

The total electric field is:

$$\vec{E}_{\text{total}} = \vec{E}_1 + \vec{E}_2 = E_0 \cos(kx - \omega t)\hat{x} + E_0 \cos(kx - \omega t + \phi)\hat{y}$$

The observed intensity I is proportional to the time-averaged square of the total electric field magnitude:

$$I \propto \langle |\vec{E}_{\text{total}}|^2 \rangle = \langle |\vec{E}_1 + \vec{E}_2|^2 \rangle = \langle |\vec{E}_1|^2 \rangle + \langle |\vec{E}_2|^2 \rangle + 2\langle \vec{E}_1 \cdot \vec{E}_2 \rangle$$

Now, evaluating the cross term:

$$\vec{E}_1 \cdot \vec{E}_2 = E_0^2 \cos(kx - \omega t) \cos(kx - \omega t + \phi)(\hat{x} \cdot \hat{y})$$

Since $\hat{x} \cdot \hat{y} = 0$ (orthogonal unit vectors), we have:

$$\vec{E}_1 \cdot \vec{E}_2 = 0$$

Therefore:

$$\langle \vec{E}_1 \cdot \vec{E}_2 \rangle = 0$$

The individual intensity terms are:

$$\langle |\vec{E}_1|^2 \rangle = \frac{E_0^2}{2}, \quad \langle |\vec{E}_2|^2 \rangle = \frac{E_0^2}{2}$$

Thus:

$$I = I_1 + I_2$$

There is no interference term, i.e., no dependence on phase difference ϕ. Therefore, no interference fringes are observed.

Conclusion: Two beams of light polarized in perpendicular directions do not interfere because their electric fields are orthogonal and yield no cross term in the intensity expression. Hence, the total intensity is simply the sum of individual intensities, and interference fringes are absent.

35 When a thin film of a transparent material is put behind one of the slits in Young's double-slit interference experiment, the zero-order fringe moves to the position previously occupied by the fourth-order bright fringe. The index of refraction of the film is $n = 1.2$ and the wavelength of light, $\lambda = 5000$ Å. Determine the thickness of the film.

Introduction: In Youngs double-slit experiment, adding a transparent film behind one slit introduces an additional optical path, shifting the interference pattern. When the central (zero-order) fringe shifts to the position of the fourth bright fringe, it indicates a path difference equivalent to 4 wavelengths. We are given:

- Refractive index of film: $n = 1.2$
- Wavelength of light in vacuum: $\lambda = 5000\,\text{Å} = 5000 \times 10^{-10}\,\text{m} = 5 \times 10^{-7}\,\text{m}$
- Fringe shift: 4 orders

We are to determine the thickness t of the film.

Solution:

The optical path difference introduced by the film is:

$$\Delta = (n - 1)t$$

This additional path causes a shift in the interference fringes. If the central fringe shifts by m orders, then:

$$(n - 1)t = m\lambda$$

Given $m = 4$, we substitute:

$$(1.2 - 1)t = 4 \cdot 5 \times 10^{-7}$$

$$0.2t = 2 \times 10^{-6}$$

$$t = \frac{2 \times 10^{-6}}{0.2} = 1 \times 10^{-5}\,\text{m} = 10\,\mu\text{m}$$

Conclusion: The thickness of the transparent film that shifts the central fringe to the position of the fourth-order bright fringe is **10 micrometers**.

36 The separation between the slits is 0.5 mm in Young's double-slit experiment. The interference pattern observed on a screen placed 5 m away reveals the location of the first maximum which is 6 mm from the centre of the pattern. Calculate the wavelength of light and separation between second and third bright fringes.

Introduction: In Youngs double-slit experiment, the interference fringes are formed on a screen placed at a distance from the slits. The position of the bright fringes is determined by the path difference of the waves from the two slits. We are given:

- Slit separation: $d = 0.5\,\text{mm} = 5 \times 10^{-4}\,\text{m}$
- Screen distance: $L = 5\,\text{m}$
- Distance of the first bright fringe from the center: $y_1 = 6\,\text{mm} = 6 \times 10^{-3}\,\text{m}$

We are to calculate:

1. The wavelength λ of the light.
2. The separation between the second and third bright fringes.

Solution:

(i) Wavelength of light:

The position of the m^{th} bright fringe is given by:

$$y_m = \frac{m\lambda L}{d}$$

For the first maximum ($m = 1$):

$$y_1 = \frac{\lambda L}{d} \Rightarrow \lambda = \frac{y_1 d}{L} = \frac{6 \times 10^{-3} \times 5 \times 10^{-4}}{5} = \frac{3 \times 10^{-6}}{5} = 6 \times 10^{-7}\,\text{m} = 600\,\text{nm}$$

(ii) Separation between second and third bright fringes:

Positions of the fringes:

$$y_2 = \frac{2\lambda L}{d}, \quad y_3 = \frac{3\lambda L}{d} \Rightarrow \Delta y = y_3 - y_2 = \frac{\lambda L}{d}$$

Substitute the known values:

$$\Delta y = \frac{6 \times 10^{-7} \times 5}{5 \times 10^{-4}} = \frac{3 \times 10^{-6}}{5 \times 10^{-4}} = 6 \times 10^{-3}\,\text{m} = 6\,\text{mm}$$

Conclusion: The wavelength of the light used in the experiment is **600 nm**, and the separation between the second and third bright fringes is **6 mm**.

37 In a Young double slit experiment, the first bright maximum is displaced by $y = 2$ cm from the central maximum. If the spacing between slits and distance from the screen are 0.1 mm and 1 m respectively, find the wavelength of light.

Introduction: In Young's double-slit experiment, the positions of bright fringes on the screen are determined by the interference condition. The fringe position for the m-th order maximum is given by the formula involving the wavelength λ, slit separation d, and screen distance L. We are given:

- Position of first bright fringe: $y = 2\,\text{cm} = 0.02\,\text{m}$
- Slit separation: $d = 0.1\,\text{mm} = 1 \times 10^{-4}\,\text{m}$
- Distance from the slits to the screen: $L = 1\,\text{m}$

We are to determine the wavelength λ of the light.

Solution:

The position of the m-th order bright fringe is given by:

$$y_m = \frac{m\lambda L}{d}$$

For the first bright maximum ($m = 1$):

$$y_1 = \frac{\lambda L}{d}$$

Solving for wavelength:

$$\lambda = \frac{y_1 d}{L}$$

Substituting the known values:

$$\lambda = \frac{0.02 \times 1 \times 10^{-4}}{1} = 2 \times 10^{-6}\,\text{m} = 2000\,\text{nm} = 2\,\mu\text{m}$$

Physical Analysis: The calculated wavelength of 2000 nm corresponds to infrared radiation, which is well outside the visible spectrum (380-700 nm). This suggests either:

- The experiment uses infrared light (which would require special detection equipment)
- There may be an error in the given measurements
- The fringe displacement might refer to a higher-order maximum rather than the first maximum

For comparison, if this were visible light (say = 600 nm), the first bright fringe would be located at:

$$y_1 = \frac{600 \times 10^{-9} \times 1}{1 \times 10^{-4}} = 6 \times 10^{-3}\,\text{m} = 6\,\text{mm}$$

Conclusion: Based on the given parameters, the calculated wavelength is **2000 nanometers (2 m)**, which corresponds to infrared radiation. This result should be verified against the experimental setup and detection method used.

38 In Michelson interferometer, 100 fringes cross the field of view when the movable mirror is displaced through 0.029 mm. Calculate the wavelength of the light source used.

Introduction: In a Michelson interferometer, when the movable mirror is displaced, the optical path difference between the two arms changes, resulting in the movement of interference fringes. Every time the optical path changes by one wavelength λ, a fringe shifts by one order. However, because the light travels to the mirror and back, a mirror displacement of Δd results in an optical path change of $2\Delta d$. We are given:

- Number of fringes counted: $N = 100$
- Mirror displacement: $\Delta d = 0.029\,\text{mm} = 2.9 \times 10^{-5}\,\text{m}$

We are to calculate the wavelength λ of the light.

Solution:

The total optical path difference created is $2\Delta d$, and this corresponds to N wavelengths:

$$2\Delta d = N\lambda \Rightarrow \lambda = \frac{2\Delta d}{N}$$

Substitute the given values:

$$\lambda = \frac{2 \times 2.9 \times 10^{-5}}{100} = \frac{5.8 \times 10^{-5}}{100} = 5.8 \times 10^{-7}\,\text{m} = 580\,\text{nm}$$

Conclusion: The wavelength of the light source used in the Michelson interferometer is **580 nanometers**.

39 Obtain the conditions for constructive interference and destructive interference in a thin film due to reflected light.

Introduction: When light is incident on a thin film, interference occurs due to the superposition of light waves reflected from the top and bottom surfaces of the film. The nature of the interference depends on the optical path difference and any phase changes upon reflection. The conditions vary depending on the refractive indices of the surrounding media.

Solution:

Consider a thin film of thickness t and refractive index n_2, with medium of refractive index n_1 above and n_3 below. For normal incidence, light reflects from both surfaces.

1. Phase changes on reflection: A phase change of π (equivalent to $\lambda/2$) occurs when light reflects from a boundary where it encounters a medium of higher refractive index:

- Reflection at top surface: phase change of π if $n_2 > n_1$, no phase change if $n_2 < n_1$
- Reflection at bottom surface: phase change of π if $n_3 > n_2$, no phase change if $n_3 < n_2$

2. Optical path difference: The optical path difference between the two reflected rays is:

$$\text{OPD} = 2n_2t$$

3. General interference conditions:

Case I: Both reflections have the same phase behavior (Either both have phase change π or both have no phase change)

Constructive interference:

$$2n_2t = m\lambda \quad \text{where } m = 1, 2, 3, \ldots$$

Destructive interference:

$$2n_2t = \left(m + \frac{1}{2}\right)\lambda \quad \text{where } m = 0, 1, 2, \ldots$$

Case II: One reflection has phase change, the other doesn't (Net phase difference of π between the two reflected rays)

Constructive interference:

$$2n_2t = \left(m + \frac{1}{2}\right)\lambda \quad \text{where } m = 0, 1, 2, \ldots$$

Destructive interference:

$$2n_2t = m\lambda \quad \text{where } m = 1, 2, 3, \ldots$$

4. Common practical examples:

Soap film in air ($n_1 = 1$, $n_2 > 1$, $n_3 = 1$): Case II applies

- Constructive: $2n_2t = \left(m + \frac{1}{2}\right)\lambda$
- Destructive: $2n_2t = m\lambda$

Oil film on water ($n_1 = 1$, $n_2 \approx 1.5$, $n_3 \approx 1.33$): Case I applies

- Constructive: $2n_2t = m\lambda$
- Destructive: $2n_2t = \left(m + \frac{1}{2}\right)\lambda$

Conclusion: The interference conditions in thin films depend critically on the relative refractive indices of the three media involved. When there is a net phase difference of π between the two reflected rays (Case II), the conditions are reversed compared to when both reflections have the same phase behavior (Case I). This explains the variety of colors observed in different thin film situations.

40 Explain with proper example the interferences due to division of wavefront and division of amplitude.

Introduction: Interference of light results from the superposition of two or more coherent light waves. The methods to produce coherent sources fall into two categories:

- Division of wavefront
- Division of amplitude

These methods are employed in different types of optical instruments to observe interference patterns. We will explain each method with clear physical examples.

Solution:

1. Division of Wavefront:

Definition: This method involves splitting a single wavefront into two or more parts which then follow different paths before overlapping to produce interference.

Example 1 Young's Double-Slit Experiment: In Young's double-slit experiment:

- A monochromatic light source illuminates a narrow single slit
- The wavefront emerging from the slit diffracts and illuminates two narrow, closely spaced slits
- These two slits act as coherent sources formed by division of the original wavefront
- The light waves from the two slits overlap on a distant screen and interfere, producing alternating bright and dark fringes

Example 2 Fresnel Double Mirror:

- Two plane mirrors are inclined at a small angle to each other
- Light from a narrow slit reflects from both mirrors, creating two coherent virtual sources
- The reflected beams overlap in a region where interference fringes are observed

Key Features:

- Suitable for extended sources
- Requires spatial coherence over the wavefront
- The coherent sources are spatially separated

2. Division of Amplitude:

Definition: In this method, a single beam is partially reflected and transmitted at one or more interfaces, creating multiple coherent beams from the same source. These beams travel different paths and interfere upon recombination.

Example 1 Michelson Interferometer: In the Michelson interferometer:

- A beam splitter divides the amplitude of the incident light into two beams
- Each beam travels to a different mirror and reflects back

- The beams recombine at the beam splitter and produce interference fringes based on the path difference
- Used for precise measurements of wavelengths and small displacements

Example 2 Thin Film Interference:

- Light incident on a thin film (soap bubble, oil film) is partially reflected at the top surface
- The remaining light enters the film and is partially reflected at the bottom surface
- The two reflected beams interfere, producing colorful patterns depending on film thickness and viewing angle

Key Features:

- More suitable for narrow, intense beams
- Requires temporal coherence (monochromaticity) of the source
- The coherent beams originate from the same point but travel different optical paths

Comparison Summary:

Division of Wavefront	**Division of Amplitude**
Spatially separates parts of a wavefront	Splits light based on partial reflection/transmission
Young's Double-Slit, Fresnel Double Mirror	Michelson Interferometer, Thin Films
Spatial coherence	Temporal coherence
Coherent sources are spatially separated	Beams originate from same point
Demonstrating basic interference	Precision measurements, optical coatings

Conclusion: Interference due to division of wavefront (e.g., Young's experiment) is achieved by geometrically splitting a single wavefront into spatially separated coherent sources, while interference due to division of amplitude (e.g., Michelson interferometer, thin films) is based on partial reflection and transmission of light beams that originate from the same point. Both methods are fundamental to various optical applications and rely on different aspects of coherence to produce stable interference patterns.