UPSC PHYSICS PYQ SOLUTION

Waves and Optics - Part 4

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50 A diffraction grating of width 5 cm with slits of width 10^{-4} cm separated by a distance of 2×10^{-4} cm is illuminated by light of wavelength 550 nm. What will be the width of the principal maximum in the diffraction pattern? Would there be any missing orders? **25**

41 What is multiple-beam interference? Discuss the advantages of multiple-beam interferometry over two-beam interferometry. Explain the fringes formed by Fabry-Perot interferometer.

Introduction

This document addresses the principles of multiple-beam interference, a fundamental concept in optics. In contrast to two-beam interference (as seen in Young's double-slit experiment), multiple-beam interference involves the superposition of numerous light waves. This typically occurs when light undergoes multiple reflections within a resonant cavity, such as a thin film or an optical instrument with parallel reflecting surfaces. The primary objective is to define multiple-beam interference, delineate its advantages over two-beam interference, and elucidate the formation of interference fringes in a Fabry-Perot interferometer, a classic example of a multiple-beam instrument. We assume the light source to be monochromatic and coherent for the formation of a stable interference pattern.

Solution

Multiple-Beam Interference

Multiple-beam interference arises from the repeated division of a light wave's amplitude, typically through successive reflections and transmissions at the surfaces of a thin film or between two parallel, highly reflective plates. Consider a beam of light incident on such a system. A fraction of the light is transmitted, and a fraction is reflected. The reflected portion strikes the second surface and is again partially reflected and transmitted. This process continues, creating a series of parallel transmitted and reflected beams. These beams are coherent as they originate from the same source. Their superposition gives rise to the interference pattern.

The intensity distribution of the transmitted light in multiple-beam interference is given by the Airy formula:

$$I_t = I_i \frac{T^2}{(1-R)^2 + 4R\sin^2(\delta/2)}$$

where:

- I_t is the transmitted intensity.
- I_i is the incident intensity.
- T is the transmittance of the surfaces.
- R is the reflectance of the surfaces.
- δ is the phase difference between successive transmitted beams.

Assuming no absorption, $T + R = 1$. The phase difference δ is given by:

$$\delta = \frac{2\pi}{\lambda}(2nd\cos\theta)$$

where:

- λ is the wavelength of the light.
- n is the refractive index of the medium between the reflecting surfaces.
- d is the separation between the surfaces.

- θ is the angle of incidence of the light within the medium.

Constructive interference (maximum intensity) occurs when $\delta = 2m\pi$, where m is an integer (the order of interference). This leads to highly intense and sharp bright fringes.

Advantages of Multiple-Beam Interferometry over Two-Beam Interferometry

Multiple-beam interferometry offers significant advantages over two-beam interferometry, primarily stemming from the sharpness of the interference fringes.

1. **Sharper Fringes:** In two-beam interference (e.g., Michelson interferometer), the intensity distribution follows a sinusoidal pattern ($I \propto \cos^2(\delta/2)$), resulting in broad fringes. In contrast, multiple-beam interference produces very sharp and narrow bright fringes against a broad dark background, especially for high reflectance ($R \to 1$). This sharpness allows for much more precise location of the fringe maxima.

2. **Higher Resolution:** The sharpness of the fringes directly translates to a higher resolving power. Instruments like the Fabry-Perot interferometer can resolve very small differences in wavelength, making them invaluable tools in spectroscopy for analyzing the fine structure of spectral lines.

3. **Improved Measurement Precision:** The narrowness of the fringes enhances the precision of measurements based on fringe displacement. A small change in the optical path difference results in a more noticeable shift of a sharp fringe compared to a broad one. This is crucial for applications like measuring the thickness of thin films or determining refractive indices with high accuracy.

Fringes Formed by a Fabry-Perot Interferometer

A Fabry-Perot interferometer consists of two parallel, flat, semi-transparent glass plates coated with a highly reflective material on their inner surfaces. The space between the plates can be air or another medium.

When an extended, monochromatic light source illuminates the interferometer, light rays enter the cavity at various angles. A single ray from the source is multiply reflected between the plates. At each reflection, a portion of the light is transmitted. A lens is used to collect and focus the parallel transmitted rays to a single point on a screen placed at the focal plane of the lens.

All rays that are transmitted through the interferometer at the same angle θ will have the same phase difference and will be focused to the same point on the screen. The condition for constructive interference for these rays is:

$$2nd\cos\theta = m\lambda$$

For a fixed plate separation d, refractive index n, and wavelength λ, this condition is satisfied for specific values of the angle θ. Since the setup has cylindrical symmetry about the axis normal to the plates, the locus of points corresponding to a constant angle of inclination θ is a circle. Therefore, the interference pattern consists of a set of concentric bright rings, each corresponding to a different order of interference m. These rings are known as "fringes of equal inclination" or Haidinger fringes. The fringes are extremely sharp and well-defined due to the multiple-beam interference effect, especially when the reflectivity of the surfaces is high.

Conclusion

Multiple-beam interference is a powerful phenomenon that occurs when a light wave is divided into many coherent beams that interfere with one another. This leads to interference patterns with significantly sharper fringes compared to those produced by two-beam interference. The primary advantages of multiple-beam interferometry are the increased sharpness of the fringes, which allows for higher resolving power and greater precision in measurements. The Fabry-Perot interferometer is a prime example of an instrument that utilizes multiple-beam interference to produce a set of sharp, concentric circular fringes, known as fringes of equal inclination. These characteristics make multiple-beam interferometry an indispensable technique in high-resolution spectroscopy and precision metrology.

42 What are the fringes of equal thickness and fringes of equal inclination? In a Newton's ring arrangement with a source emitting two wavelengths $\lambda_1 = 6 \times 10^{-7}$ m and $\lambda_2 = 5.9 \times 10^{-7}$ m, it is found that the m^{th} dark ring due to one wavelength coincides with the $(m+1)^{\text{th}}$ dark ring due to the other. Find the diameter of the m^{th} dark ring, if the radius of curvature of the lens is 90 cm.

Introduction

This problem consists of two parts. The first part requires a definition of two types of interference fringes observed in thin films: fringes of equal thickness and fringes of equal inclination. The second part is a numerical problem based on the Newton's rings experiment, which is an example of fringes of equal thickness.

In the numerical problem, we are given a Newton's rings setup illuminated by a light source with two distinct wavelengths, $\lambda_1 = 6 \times 10^{-7}\,\text{m}$ and $\lambda_2 = 5.9 \times 10^{-7}\,\text{m}$. We are told that the m^{th} dark ring for λ_1 coincides with the $(m+1)^{\text{th}}$ dark ring for λ_2. The radius of curvature of the plano-convex lens is $R = 90\,\text{cm} = 0.9\,\text{m}$. We need to find the order of the ring, m, and the diameter of this m^{th} dark ring (D_m). We assume the interference occurs in an air film ($\mu \approx 1$) and is observed in the reflected light under normal incidence.

Solution

Part 1: Fringes of Equal Thickness and Equal Inclination

Interference fringes in thin films arise from the superposition of light waves reflected from the top and bottom surfaces of the film. The nature of the fringes depends on the geometry of the film and the nature of the light source.

1. **Fringes of Equal Thickness (Fizeau Fringes)**: These fringes are observed when a thin film has a varying thickness, and it is illuminated by a broad source of monochromatic light. Each fringe represents a locus of points where the thickness of the film is constant. The path difference between the interfering rays depends primarily on the film's thickness (t) at that point. A classic example is the interference pattern produced by a wedge-shaped air film or the Newton's rings experiment, where the thickness of the air film between the lens and the glass plate is constant along a circle centered on the point of contact.

2. **Fringes of Equal Inclination (Haidinger Fringes)**: These fringes are produced when a thin film of uniform thickness is illuminated by a broad source of light. In this case, the path difference between interfering rays depends only on the angle of inclination (θ) at which the light strikes the film. Each fringe is formed by rays that have the same angle of inclination. These fringes are typically observed at infinity or in the focal plane of a convex lens. An example is the interference pattern observed in a Michelson interferometer when its mirrors are perfectly parallel.

Part 2: Newton's Rings Calculation

In a Newton's rings experiment, for reflection at normal incidence, a dark fringe (destructive interference) occurs when the optical path difference is an integer multiple of the wavelength. Due to a phase shift of π (equivalent to a path difference of $\lambda/2$) upon reflection at the denser medium (the glass plate), the condition for the

k^{th} dark ring is:

$$2\mu t = k\lambda$$

For an air film, $\mu = 1$, and we denote the order by m, so the condition for the m^{th} dark ring is:

$$2t = m\lambda \quad (m = 0, 1, 2, ...)$$

The relationship between the thickness of the air film t, the radius of the ring r_m, and the radius of curvature of the lens R is given by the geometric relation $r_m^2 \approx 2Rt$. Substituting $t = r_m^2/(2R)$ into the condition for a dark fringe, we get:

$$2\left(\frac{r_m^2}{2R}\right) = m\lambda \implies r_m^2 = m\lambda R$$

The diameter of the m^{th} dark ring, $D_m = 2r_m$, is therefore given by:

$$D_m^2 = 4r_m^2 = 4m\lambda R$$

Now, we apply the given condition that the m^{th} dark ring for λ_1 coincides with the $(m+1)^{\text{th}}$ dark ring for λ_2.

$$D_{m,\lambda_1} = D_{m+1,\lambda_2}$$

Squaring both sides gives:

$$D_{m,\lambda_1}^2 = D_{m+1,\lambda_2}^2$$

Using the formula $D^2 = 4m\lambda R$:

$$4m\lambda_1 R = 4(m+1)\lambda_2 R$$

The term $4R$ cancels out:

$$m\lambda_1 = (m+1)\lambda_2$$

$$m\lambda_1 = m\lambda_2 + \lambda_2$$

$$m(\lambda_1 - \lambda_2) = \lambda_2$$

Now, we can solve for m by substituting the given values for the wavelengths:

$$\lambda_1 = 6 \times 10^{-7}\,\text{m}$$

$$\lambda_2 = 5.9 \times 10^{-7}\,\text{m}$$

$$m = \frac{\lambda_2}{\lambda_1 - \lambda_2} = \frac{5.9 \times 10^{-7}\,\text{m}}{(6 \times 10^{-7} - 5.9 \times 10^{-7})\,\text{m}} = \frac{5.9 \times 10^{-7}}{0.1 \times 10^{-7}} = 59$$

So, the coincidence occurs for the 59^{th} dark ring of wavelength λ_1 and the 60^{th} dark ring of wavelength λ_2.

The question asks for the diameter of the m^{th} dark ring, which is D_{59} for λ_1.

$$D_{59}^2 = 4m\lambda_1 R$$

Substituting the values $m = 59$, $\lambda_1 = 6 \times 10^{-7}\,\text{m}$, and $R = 0.9\,\text{m}$:

$$D_{59}^2 = 4 \times 59 \times (6 \times 10^{-7}\,\text{m}) \times (0.9\,\text{m})$$

$$D_{59}^2 = 236 \times 5.4 \times 10^{-7}\,\text{m}^2$$

$$D_{59}^2 = 1274.4 \times 10^{-7}\,\text{m}^2 = 1.2744 \times 10^{-4}\,\text{m}^2$$

Taking the square root to find the diameter:

$$D_{59} = \sqrt{1.2744 \times 10^{-4}}\,\text{m} = 1.12889 \times 10^{-2}\,\text{m}$$

Converting to centimeters:

$$D_{59} \approx 1.129\,\mathrm{cm}$$

Conclusion

The two types of interference fringes are defined as follows:

- **Fringes of equal thickness** are loci of points of constant film thickness.
- **Fringes of equal inclination** are loci of points corresponding to a constant angle of incidence of light.

From the given conditions for the Newton's rings experiment, the order of the dark ring for the first wavelength is found to be $m = 59$. The diameter of this 59^{th} dark ring is calculated to be approximately $1.129\,\mathrm{cm}$. This result is physically reasonable for a standard Newton's rings apparatus.

43 What are Newton's rings? How are they formed by two curved surfaces?

Introduction

Newton's rings are a classic interference phenomenon, first studied by Sir Isaac Newton, that manifests as a series of concentric, alternating bright and dark circular fringes. They are a prime example of thin-film interference. The pattern arises from the reflection of light between two surfacestypically a plano-convex lens and a flat glass platebut the principle extends to any two curved surfaces placed in contact, creating a thin film of air between them.

Solution

1. Physical Mechanism and Path Difference

When a beam of monochromatic light of wavelength λ is normally incident on the setup, it encounters a thin film of air (refractive index $\mu \approx 1$) of varying thickness t.

- A portion of the light reflects from the lower surface of the upper lens (a glass-to-air interface, which is a denser-to-rarer medium reflection). This reflection occurs with **no phase change**.
- Another portion of the light transmits through the air film and reflects from the upper surface of the lower lens (an air-to-glass interface, which is a rarer-to-denser medium reflection). This reflection introduces a **phase change of π radians**, which is equivalent to an extra path length of $\lambda/2$.

The two reflected rays are coherent and interfere. The total optical path difference (Δ) between them is the sum of the geometric path difference (twice the film thickness for normal incidence, $2t$) and the path difference from the phase change.

$$\Delta = 2t + \frac{\lambda}{2}$$

2. Geometry of the Air Film between Two Curved Surfaces

Let two spherical surfaces with radii of curvature R_1 and R_2 be in contact. Using the property of a circle, the thickness t of the air film at a radial distance r from the point of contact is given by the sum or difference of the individual sagittas (t_1 and t_2). For a single surface of radius R, the sagitta t is approximated as $t \approx \frac{r^2}{2R}$. Thus, the total thickness is $t = \frac{r^2}{2}\left(\frac{1}{R_1} \pm \frac{1}{R_2}\right)$.

We can define an effective radius of curvature, R_{eff}, such that $t = \frac{r^2}{2R_{eff}}$. The value of $1/R_{eff}$ depends on the configuration:

- For two convex surfaces: $\frac{1}{R_{eff}} = \frac{1}{R_1} + \frac{1}{R_2}$
- For a convex surface (R_1) on a concave surface (R_2): $\frac{1}{R_{eff}} = \left|\frac{1}{R_1} - \frac{1}{R_2}\right|$

Substituting the expression for t into the path difference equation gives:

$$\Delta = 2\left(\frac{r^2}{2R_{eff}}\right) + \frac{\lambda}{2} = \frac{r^2}{R_{eff}} + \frac{\lambda}{2}$$

3. Conditions for Interference and Ring Radii

Constructive Interference (Bright Rings): For the reflected rays to interfere constructively, the path difference must be an integer multiple of the wavelength. Condition: $\Delta = m\lambda$, where $m = 1, 2, 3, \ldots$

$$\frac{r_m^2}{R_{eff}} + \frac{\lambda}{2} = m\lambda$$

$$\frac{r_m^2}{R_{eff}} = \left(m - \frac{1}{2}\right)\lambda = \frac{(2m-1)\lambda}{2}$$

The radius of the m-th bright ring is:

$$r_m = \sqrt{\frac{(2m-1)\lambda R_{eff}}{2}}$$

Destructive Interference (Dark Rings): For destructive interference, the path difference must be a half-integer multiple of the wavelength. Condition: $\Delta = (m + \frac{1}{2})\lambda$, where $m = 0, 1, 2, \ldots$

$$\frac{r_m^2}{R_{eff}} + \frac{\lambda}{2} = \left(m + \frac{1}{2}\right)\lambda$$

$$\frac{r_m^2}{R_{eff}} = m\lambda$$

The radius of the m-th dark ring is:

$$r_m = \sqrt{m\lambda R_{eff}}$$

At the center of the pattern (the point of contact), the thickness $t = 0$. For $m = 0$ in the dark ring condition, we get $r_0 = 0$. The path difference here is $\Delta = 0 + \lambda/2$, which satisfies the condition for destructive interference. Therefore, the central spot is always dark.

Conclusion

Newton's rings formed by two curved surfaces are a direct consequence of thin-film interference in the wedge-shaped air film between them. The radii of the rings depend on the wavelength of light (λ) and the effective radius of curvature (R_{eff}) of the setup. The radius of the m-th bright ring is given by $r_m = \sqrt{(2m-1)\lambda R_{eff}/2}$, and the radius of the m-th dark ring is $r_m = \sqrt{m\lambda R_{eff}}$. The characteristic dark center is a key feature, resulting from the π phase shift upon reflection at the denser medium.

44 Discuss the conditions for interference. Describe Young's double-slit experiment and derive an expression for the estimation of fringe width. Discuss its dependency on various parameters. Green light of wavelength 5100 Å from a narrow slit is incident on a double-slit. If the overall separation of 10 fringes on a screen 200 cm away is 2 cm, find the slit separation.

Introduction

This problem requires a comprehensive discussion of optical interference, including the necessary conditions for producing stable interference patterns, a detailed description of Young's double-slit experiment (YDSE), derivation of the fringe width expression, analysis of parameter dependencies, and solution of a numerical problem.

Given parameters for the numerical problem:

- Wavelength of light: $\lambda = 5100\,\text{Å} = 5100 \times 10^{-10}\,\text{m} = 5.1 \times 10^{-7}\,\text{m}$
- Screen distance: $D = 200\,\text{cm} = 2.0\,\text{m}$
- Total width of 10 fringes: $10\beta = 2\,\text{cm} = 0.02\,\text{m}$

Solution

1. Conditions for Sustained Interference of Light

For two light waves to produce a stable, observable interference pattern, the following conditions must be satisfied:

1. **Coherence:** The sources must be coherent, meaning they maintain a constant phase relationship. This requires:
   - **Temporal coherence:** The waves must have the same frequency (monochromatic light)
   - **Spatial coherence:** The sources should be sufficiently small and close together
2. **Constant Phase Difference:** The phase difference between the interfering waves must remain constant over the observation time.
3. **Comparable Amplitudes:** The interfering waves should have similar amplitudes for maximum fringe visibility. The visibility $V = \frac{I_{max}-I_{min}}{I_{max}+I_{min}}$ is maximized when amplitudes are equal.
4. **Same Polarization:** The interfering waves must have the same polarization state, as waves with perpendicular polarizations do not interfere.
5. **Appropriate Geometry:** The path difference between interfering rays should be of the order of a few wavelengths for observable effects.

2. Young's Double-Slit Experiment (YDSE)

Thomas Young's 1801 experiment provided the first definitive proof of light's wave nature:

Experimental Setup:

- A narrow slit S is illuminated by monochromatic light
- This primary slit creates cylindrical wavefronts that illuminate two parallel narrow slits S_1 and S_2
- The slits S_1 and S_2 are separated by distance d and equidistant from S
- These act as coherent secondary sources, producing interfering cylindrical waves
- A screen is placed at distance D from the double slit to observe the interference pattern

3. Derivation of Fringe Width

Consider point P on the screen at distance y from the central axis O. Let the path lengths from S_1 and S_2 to point P be r_1 and r_2 respectively.

Using geometry:

$$r_1 = \sqrt{D^2 + \left(y - \frac{d}{2}\right)^2}$$

$$r_2 = \sqrt{D^2 + \left(y + \frac{d}{2}\right)^2}$$

The path difference is:

$$\Delta = r_2 - r_1$$

For the approximation $D \gg d$ and $D \gg y$, we can expand using the binomial theorem:

$$r_1 \approx D + \frac{\left(y - \frac{d}{2}\right)^2}{2D} \approx D + \frac{y^2}{2D} - \frac{yd}{2D}$$

$$r_2 \approx D + \frac{\left(y + \frac{d}{2}\right)^2}{2D} \approx D + \frac{y^2}{2D} + \frac{yd}{2D}$$

Therefore, the path difference becomes:

$$\Delta = r_2 - r_1 \approx \frac{yd}{D}$$

Interference Conditions:

Constructive Interference (Bright Fringes):

$$\Delta = n\lambda, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$\frac{y_n d}{D} = n\lambda$$

$$y_n = \frac{n\lambda D}{d}$$

Destructive Interference (Dark Fringes):

$$\Delta = \left(n + \frac{1}{2}\right)\lambda, \quad n = 0, \pm 1, \pm 2, \ldots$$

$$y'_n = \frac{\left(n + \frac{1}{2}\right)\lambda D}{d}$$

Fringe Width: The fringe width β is the distance between consecutive bright (or dark) fringes:

$$\beta = y_{n+1} - y_n = \frac{(n+1)\lambda D}{d} - \frac{n\lambda D}{d}$$

$$\boxed{\beta = \frac{\lambda D}{d}}$$

4. Parameter Dependencies

The fringe width formula $\beta = \frac{\lambda D}{d}$ shows:

1. **Wavelength dependence:** $\beta \propto \lambda$
   - Red light produces wider fringes than blue light
   - For white light, different colors produce different fringe widths, leading to colored fringes
2. **Screen distance dependence:** $\beta \propto D$
   - Moving the screen farther increases fringe separation
   - Linear relationship allows easy scaling
3. **Slit separation dependence:** $\beta \propto \frac{1}{d}$
   - Closer slits produce wider fringes
   - This inverse relationship is crucial for fringe visibility

5. Numerical Solution

Given data:

- $\lambda = 5.1 \times 10^{-7}\,\mathrm{m}$
- $D = 2.0\,\mathrm{m}$
- Total separation of 10 fringes $= 0.02\,\mathrm{m}$

First, calculate the fringe width:

$$\beta = \frac{0.02\,\mathrm{m}}{10} = 2.0 \times 10^{-3}\,\mathrm{m}$$

Using the fringe width formula:

$$\beta = \frac{\lambda D}{d}$$

Solving for slit separation:

$$d = \frac{\lambda D}{\beta} = \frac{(5.1 \times 10^{-7}\,\mathrm{m})(2.0\,\mathrm{m})}{2.0 \times 10^{-3}\,\mathrm{m}}$$

$$d = \frac{1.02 \times 10^{-6}}{2.0 \times 10^{-3}} = 5.1 \times 10^{-4}\,\mathrm{m}$$

$$\boxed{d = 0.51\,\mathrm{mm}}$$

Conclusion

Young's double-slit experiment elegantly demonstrates wave interference and provides quantitative relationships between experimental parameters. The fringe width formula $\beta = \frac{\lambda D}{d}$ establishes fundamental scaling laws for interference patterns. For the given problem, the calculated slit separation of 0.51 mm demonstrates the precision achievable with interference-based measurements and validates the wave theory of light.

45 Newton's rings are observed between a spherical surface of radius of curvature 100 cm and a plane glass plate. The diameters of 4th and 15th bright rings are 0.314 cm and 0.574 cm, respectively. Calculate the diameters of 24th and 36th bright rings and also the wavelength of light used.

Introduction

This problem involves interference in the thin air film formed between a plano-convex lens and a plane glass plate, producing Newton's rings. We analyze the system in reflected light to determine the wavelength and predict ring diameters.

Given Data:

- Radius of curvature: $R = 100\,\text{cm}$
- Diameter of 4th bright ring: $D_4 = 0.314\,\text{cm}$
- Diameter of 15th bright ring: $D_{15} = 0.574\,\text{cm}$

Required:

- Wavelength of light: λ
- Diameter of 24th bright ring: D_{24}
- Diameter of 36th bright ring: D_{36}

Solution

1. Theoretical Framework

For Newton's rings in reflected light, a phase change of π occurs at the air-glass interface. The condition for the n-th bright ring is:

$$2t = (2n-1)\frac{\lambda}{2}$$

where t is the air film thickness at radius r_n.

From geometry, for small thickness: $t \approx \frac{r_n^2}{2R}$

Substituting and using $D_n = 2r_n$:

$$D_n^2 = 2(2n-1)R\lambda$$

For two different rings ($m > n$):

$$D_m^2 - D_n^2 = 4(m-n)R\lambda$$

2. Wavelength Calculation

Using the 15th and 4th rings:

$$D_{15}^2 = (0.574)^2 = 0.329476\,\text{cm}^2$$

$$D_4^2 = (0.314)^2 = 0.098596\,\text{cm}^2$$

$$D_{15}^2 - D_4^2 = 0.230880\,\text{cm}^2$$

From the difference formula:

$$\lambda = \frac{D_{15}^2 - D_4^2}{4(15-4)R} = \frac{0.230880}{4 \times 11 \times 100} = \frac{0.230880}{4400}$$

$$\lambda = 5.247 \times 10^{-5}\,\text{cm} = 5247\,\text{Å}$$

3. Verification Let's verify our result is reasonable. The calculated wavelength (5247 Å) corresponds to green light, which is physically reasonable.

4. Ring Diameter Calculations

Using the proportionality relationship to minimize rounding errors:

For the 24th ring:

$$\frac{D_{24}^2 - D_4^2}{D_{15}^2 - D_4^2} = \frac{24-4}{15-4} = \frac{20}{11}$$

$$D_{24}^2 = D_4^2 + \frac{20}{11}(D_{15}^2 - D_4^2)$$

$$D_{24}^2 = 0.098596 + \frac{20}{11}(0.230880) = 0.098596 + 0.419782 = 0.518378\,\text{cm}^2$$

$$D_{24} = \sqrt{0.518378} = 0.720\,\text{cm}$$

For the 36th ring:

$$\frac{D_{36}^2 - D_4^2}{D_{15}^2 - D_4^2} = \frac{36-4}{15-4} = \frac{32}{11}$$

$$D_{36}^2 = D_4^2 + \frac{32}{11}(D_{15}^2 - D_4^2)$$

$$D_{36}^2 = 0.098596 + \frac{32}{11}(0.230880) = 0.098596 + 0.671651 = 0.770247\,\text{cm}^2$$

$$D_{36} = \sqrt{0.770247} = 0.878\,\text{cm}$$

5. Cross-verification We can verify our results using the fundamental equation:

- For $n = 24$: $D_{24}^2 = 2(2 \times 24 - 1) \times 100 \times 5.247 \times 10^{-5} = 0.518\,\text{cm}^2$
- For $n = 36$: $D_{36}^2 = 2(2 \times 36 - 1) \times 100 \times 5.247 \times 10^{-5} = 0.770\,\text{cm}^2$

Final Results

$$\boxed{\begin{aligned} \text{Wavelength of light: } & \lambda = 5247\,\text{Å} \\ \text{Diameter of 24th bright ring: } & D_{24} = 0.720\,\text{cm} \\ \text{Diameter of 36th bright ring: } & D_{36} = 0.878\,\text{cm} \end{aligned}}$$

Physical Interpretation The results confirm that the square of ring diameters increases linearly with ring order, as predicted by theory. The calculated wavelength corresponds to green light in the visible spectrum, which is consistent with typical Newton's ring experiments.

46 Obtain the expression for the primary focal length of Fresnel zone plate.

Introduction

A Fresnel zone plate is a diffractive optical element used to focus light or other waves. Unlike a conventional lens, which operates by refraction, a zone plate uses diffraction. It consists of a set of concentric annular rings, known as Fresnel zones, which are alternately transparent and opaque to the incident radiation. The design is based on Augustin-Jean Fresnel's method of dividing a wavefront into zones, called half-period zones.

The working principle of a zone plate is to block the light from every other half-period zone (e.g., the even-numbered zones). The light waves passing through the remaining transparent zones (the odd-numbered zones) interfere constructively at a specific point on the axis, called the focus. This constructive interference results in a high-intensity spot. A zone plate has multiple foci, but the most intense one is called the primary focus. This response derives the expression for the primary focal length of a Fresnel zone plate.

Solution

Let us consider a plane wavefront of monochromatic light with wavelength λ incident normally upon a Fresnel zone plate. Let the plate be situated in the XY-plane. We wish to find the position of the primary focus, P, which lies on the z-axis (the axis of symmetry). Let the distance from the center of the zone plate, O, to the point P be f_p (the primary focal length).

According to the principle of a Fresnel zone plate, the radii of the zones are chosen such that the path difference between a wave diffracted from the edge of the n-th zone and a wave passing through the center of the plate is an integer multiple of a half-wavelength.

The path of the ray from the center of the plate O to the focal point P is simply f_p. The path of a ray from the edge of the n-th circular zone (with radius r_n) to the point P can be found using the Pythagorean theorem. This path length is $\sqrt{f_p^2 + r_n^2}$.

For the waves from the edge of the n-th zone and the center to arrive at P with maximum constructive interference between successive transparent zones, the path difference between them must be an odd multiple of $\lambda/2$. For the n-th zone edge, this condition is:

$$\sqrt{f_p^2 + r_n^2} - f_p = n\frac{\lambda}{2}$$

Here, n is the integer representing the zone number ($n = 1, 2, 3, \ldots$).

To solve for the primary focal length f_p, we rearrange the equation:

$$\sqrt{f_p^2 + r_n^2} = f_p + n\frac{\lambda}{2}$$

Squaring both sides of the equation, we get:

$$f_p^2 + r_n^2 = \left(f_p + n\frac{\lambda}{2}\right)^2$$

$$f_p^2 + r_n^2 = f_p^2 + 2f_p\left(n\frac{\lambda}{2}\right) + \left(n\frac{\lambda}{2}\right)^2$$

$$f_p^2 + r_n^2 = f_p^2 + f_p n\lambda + \frac{n^2\lambda^2}{4}$$

By canceling the f_p^2 term from both sides, we are left with:

$$r_n^2 = f_p n\lambda + \frac{n^2\lambda^2}{4}$$

In practical optical scenarios, the focal length f_p is much larger than the wavelength of light λ (typically $f_p \sim$ mm to cm, while $\lambda \sim$ hundreds of nm). For reasonable zone numbers, the condition $f_p \gg n\lambda/4$ holds, making the term $\frac{n^2\lambda^2}{4}$ negligible compared to $f_p n\lambda$. We can therefore make the approximation:

$$r_n^2 \approx f_p n\lambda$$

Solving for the primary focal length:

$$f_p = \frac{r_n^2}{n\lambda}$$

This is the general expression for the primary focal length. For the first zone ($n = 1$), this simplifies to the most commonly used form:

$$f_p = \frac{r_1^2}{\lambda}$$

where r_1 is the radius of the first (innermost) Fresnel zone.

Conclusion

The expression for the primary focal length, f_p, of a Fresnel zone plate is:

$$f_p = \frac{r_1^2}{\lambda}$$

where r_1 is the radius of the first zone and λ is the wavelength of the incident light. More generally, for any zone n:

$$f_p = \frac{r_n^2}{n\lambda}$$

This formula highlights a key characteristic of a zone plate: its focal length is inversely proportional to the wavelength of light. This is in stark contrast to a conventional refractive lens, whose focal length is generally directly proportional to the wavelength. This property makes zone plates highly useful as focusing elements in applications where traditional lenses are impractical, such as for X-rays and other forms of short-wavelength radiation.

47 The Fraunhofer single-slit diffraction intensity is given by $I = I_0 \frac{\sin^2 x}{x^2}$, where $x = \frac{\pi d y}{\lambda l}$ with l as distance from slit to source, d the slit width, y the detector distance, and λ the wavelength. What is the value of cumulative intensity $\int_{-\infty}^{\infty} I(y)dy$?

Introduction

This problem asks for the evaluation of the definite integral of the Fraunhofer single-slit diffraction intensity function over all possible detector positions y. The intensity distribution is given by:

$$I(y) = I_0 \frac{\sin^2 x}{x^2}$$

where $x = \frac{\pi d y}{\lambda l}$, and we need to find:

$$\int_{-\infty}^{\infty} I(y)\, dy$$

Note: While the problem states l as "distance from slit to source," in the context of Fraunhofer diffraction with a detector at distance y, this parameter l effectively represents the characteristic distance in the diffraction geometry.

Solution

We set up the integral:

$$\int_{-\infty}^{\infty} I_0 \frac{\sin^2\left(\frac{\pi d y}{\lambda l}\right)}{\left(\frac{\pi d y}{\lambda l}\right)^2}\, dy$$

To evaluate this integral, we use the substitution $x = \frac{\pi d y}{\lambda l}$.

From this substitution:

$$dx = \frac{\pi d}{\lambda l} dy$$

$$dy = \frac{\lambda l}{\pi d} dx$$

The limits of integration remain from $-\infty$ to $+\infty$ since both x and y range over the same infinite domain.

Substituting into the integral:

$$\int_{-\infty}^{\infty} I_0 \frac{\sin^2 x}{x^2} \cdot \frac{\lambda l}{\pi d}\, dx$$

Factoring out the constants:

$$I_0 \frac{\lambda l}{\pi d} \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$$

The integral $\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx$ is a standard mathematical result that can be evaluated using various methods including:

- Contour integration in complex analysis

- Parseval's theorem from Fourier analysis
- Integration by parts combined with the Dirichlet integral

The value of this integral is:

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\, dx = \pi$$

Substituting this result:

$$\int_{-\infty}^{\infty} I(y)\, dy = I_0 \frac{\lambda l}{\pi d} \cdot \pi = I_0 \frac{\lambda l}{d}$$

Conclusion

The value of the cumulative intensity integral is:

$$\int_{-\infty}^{\infty} I(y)\, dy = I_0 \frac{\lambda l}{d}$$

This result represents the mathematical evaluation of the given integral. The expression shows that the integral value is proportional to the peak intensity I_0, the wavelength λ, and the distance parameter l, while being inversely proportional to the slit width d. This is purely a mathematical result of integrating the given intensity function over all detector positions.

48 In relation to a plane diffraction grating having 5000 lines per cm and irradiated by light of wavelength 6000 Å, answer the following: (i) What is the highest order spectrum which may be observed? (ii) If the width of opaque space is exactly twice that of transparent space, which order of spectra will be absent?

Introduction: We analyze a diffraction grating with 5000 lines per cm illuminated by light of wavelength $\lambda = 6000\,\text{Å} = 6 \times 10^{-7}\,\text{m}$. We need to find:

(i) The highest observable diffraction order

(ii) Which orders are absent due to the specific slit-to-opaque width ratio

Given Data:

- Lines per unit length: $N = 5000\,\text{lines/cm} = 5 \times 10^5\,\text{lines/m}$
- Wavelength: $\lambda = 6000\,\text{Å} = 6 \times 10^{-7}\,\text{m}$
- Opaque width = 2 Œ transparent width

Solution:

Part (i): Maximum Observable Order

The grating spacing (distance between adjacent slits) is:

$$d = \frac{1}{N} = \frac{1}{5 \times 10^5} = 2 \times 10^{-6}\,\text{m}$$

The grating equation for constructive interference is:

$$d \sin\theta = n\lambda$$

For the maximum observable order, we set $\sin\theta = 1$ (grazing angle):

$$n_{\text{max}} = \frac{d}{\lambda} = \frac{2 \times 10^{-6}}{6 \times 10^{-7}} = \frac{20}{6} = 3.33$$

Since n must be an integer:

$$n_{\text{max}} = \lfloor 3.33 \rfloor = 3$$

Part (ii): Missing Orders Due to Slit Structure

Let the transparent slit width be a and the opaque width be $b = 2a$.

The total grating spacing is:

$$d = a + b = a + 2a = 3a$$

Therefore: $a = \frac{d}{3} = \frac{2\times10^{-6}}{3}\,\text{m}$

Physical Mechanism: Missing orders occur when the single-slit diffraction envelope has minima that coincide with grating maxima. The single-slit diffraction minima occur at:

$$a \sin\theta = m\lambda \quad \text{for } m = 1, 2, 3, \ldots$$

The grating maxima occur at:

$$d \sin\theta = n\lambda$$

For missing orders, these conditions must be satisfied simultaneously:

$$\frac{a \sin\theta}{\lambda} = m \quad \text{and} \quad \frac{d \sin\theta}{\lambda} = n$$

This gives us:

$$\frac{n}{m} = \frac{d}{a} = \frac{3a}{a} = 3$$

Therefore: $n = 3m$

The missing orders are those where n is a multiple of 3:

$$n_{\text{missing}} = 3, 6, 9, 12, \ldots$$

Complete Analysis: Given that the maximum observable order is $n_{\max} = 3$, we need to check which orders are both theoretically possible and not eliminated by the single-slit envelope:

- Possible orders: $n = 0, 1, 2, 3$
- Missing orders due to slit structure: $n = 3, 6, 9, \ldots$
- Orders that would be missing within the observable range: $n = 3$

Conclusion:

(i) The highest observable diffraction order is $\boxed{n = 3}$.

(ii) Within the observable range, the $\boxed{3^{\text{rd}} \text{ order}}$ spectrum will be absent due to the single-slit diffraction envelope. The observable orders are $n = 0, 1, 2$ only.

Physical Interpretation: The absence of the 3rd order occurs because the single-slit diffraction pattern (determined by the individual slit width a) modulates the overall grating pattern. When the opaque region is twice the transparent region, the single-slit diffraction minimum coincides exactly with what would be the 3rd-order grating maximum, causing its suppression.

49 Distinguish between Fresnel and Fraunhofer classes of diffraction. Show that the area of each Fresnel half-period zone is same.

Introduction: Diffraction phenomena are categorized into two primary types based on the geometry of wavefronts and the positions of the source and observation screen: Fresnel and Fraunhofer diffraction. In this problem, we are to distinguish between these two classes and then analytically show that the area of each Fresnel half-period zone is the same.

Solution:

Distinction between Fresnel and Fraunhofer Diffraction:

(i) **Wavefront Geometry:**

- **Fresnel Diffraction:** The source or the screen or both are at finite distances; hence, the incident wavefront is either spherical or cylindrical.
- **Fraunhofer Diffraction:** The source and the screen are effectively at infinite distances (or lenses are used to simulate this), and the incident wavefront is plane.

(ii) **Nature of Analysis:**

- **Fresnel Diffraction:** Involves complex integration due to varying path differences across the aperture.
- **Fraunhofer Diffraction:** Involves simpler Fourier transform analysis of aperture functions.

(iii) **Experimental Setup:**

- **Fresnel Diffraction:** No lenses are required.
- **Fraunhofer Diffraction:** Requires collimating and focusing lenses.

(iv) **Practical Use:**

- **Fresnel Diffraction:** Used in near-field applications like edge diffraction and zone plates.
- **Fraunhofer Diffraction:** Used in far-field analysis like diffraction by slits, gratings.

Proof that the Area of Each Fresnel Half-Period Zone is Same:

Let a plane wave be incident on a spherical surface centered at the observation point P located at a distance b from the wavefront.

The radius r_n of the n-th Fresnel half-period zone is defined such that the path difference between r_n and r_{n-1} is $\lambda/2$.

The path difference between a point at radius r on the wavefront and the center is approximately:

$$\delta = \frac{r^2}{2b}$$

Set the condition that the difference in δ between two successive zones equals $\lambda/2$:

$$\frac{r_n^2 - r_{n-1}^2}{2b} = \frac{\lambda}{2} \Rightarrow r_n^2 - r_{n-1}^2 = b\lambda$$

Define the area A_n of the n-th half-period zone as:

$$A_n = \pi r_n^2 - \pi r_{n-1}^2 = \pi(r_n^2 - r_{n-1}^2) = \pi b\lambda$$

This result is independent of n, hence the area of each half-period zone is the same and equal to $\pi b\lambda$.

Conclusion: Fresnel and Fraunhofer diffraction differ fundamentally in geometry and setup. In Fresnel diffraction, the area of each half-period zone is constant and equals $\pi b\lambda$, where b is the distance to the observation point and λ is the wavelength of light.

50 A diffraction grating of width 5 cm with slits of width 10^{-4} cm separated by a distance of 2×10^{-4} cm is illuminated by light of wavelength 550 nm. What will be the width of the principal maximum in the diffraction pattern? Would there be any missing orders?

Introduction: A diffraction grating has a total width of $W = 5\,\text{cm}$ and comprises slits each of width $a = 10^{-4}$ cm, with an adjacent slit separation (grating element) of $d = 2 \times 10^{-4}$ cm. The wavelength of the incident light is $\lambda = 550\,\text{nm} = 5.5 \times 10^{-5}$ cm. We are to determine the angular width of the principal maximum and check for any missing orders due to the relationship between a and d.

Solution:

(i) Width of the Principal Maximum:

The total number of slits N in the grating is:

$$N = \frac{W}{d} = \frac{5}{2 \times 10^{-4}} = 2.5 \times 10^4$$

For a diffraction grating, the angular width of a principal maximum is determined by the angular separation between the first minima on either side of the maximum. The condition for minima adjacent to the n^{th} order principal maximum is:

$$Nd \sin\theta = nN\lambda \pm \lambda$$

For the central maximum ($n = 0$), the first minima occur at:

$$Nd \sin\theta = \pm\lambda$$

$$\sin\theta = \pm\frac{\lambda}{Nd}$$

Substituting the values:

$$\sin\theta = \pm\frac{5.5 \times 10^{-5}}{2.5 \times 10^4 \times 2 \times 10^{-4}} = \pm\frac{5.5 \times 10^{-5}}{5} = \pm 1.1 \times 10^{-5}$$

Since $\sin\theta \ll 1$, we can use the small angle approximation $\sin\theta \approx \theta$:

$$\theta \approx \pm 1.1 \times 10^{-5} \text{ radians}$$

Converting to degrees:

$$\theta \approx \pm 1.1 \times 10^{-5} \times \frac{180}{\pi} \approx \pm 6.3 \times 10^{-4} \text{ degrees}$$

The total angular width of the central principal maximum is:

$$\Delta\theta = 2 \times 6.3 \times 10^{-4} = 1.26 \times 10^{-3} \text{ degrees} = 4.54 \text{ arcseconds}$$

(ii) Missing Orders:

Missing orders occur when a principal maximum of the grating coincides with a minimum of the single-slit diffraction envelope. The conditions are:

- Principal maxima: $d \sin\theta = n\lambda$
- Single-slit minima: $a \sin\theta = m\lambda$ (where $m = \pm1, \pm2, \ldots$)

For coincidence:

$$\frac{d \sin\theta}{a \sin\theta} = \frac{n\lambda}{m\lambda} \Rightarrow \frac{d}{a} = \frac{n}{m}$$

With $d = 2 \times 10^{-4}$ cm and $a = 10^{-4}$ cm:

$$\frac{d}{a} = \frac{2 \times 10^{-4}}{10^{-4}} = 2$$

Therefore: $\frac{n}{m} = 2$, which gives $n = 2m$.

This means the orders $n = 2, 4, 6, 8, \ldots$ (all even orders) will be missing because they coincide with the minima of the single-slit diffraction pattern.

Conclusion: The angular width of the central principal maximum is approximately 1.26×10^{-3} degrees or 4.54 arcseconds. Due to the relation $d = 2a$, all even-order diffraction maxima (2nd, 4th, 6th, ...) are missing in the diffraction pattern.