UPSC PHYSICS PYQ SOLUTION
Waves and Optics - Part 6

Contents

51 A parallel beam of light from a He-Ne laser ($\lambda = 630$ nm) is made to fall on a narrow slit of width 0.2×10^{-3} m. The Fraunhofer diffraction pattern is observed on a screen placed in the focal plane of a convex lens of focal length 0.3 m. Calculate the distance between (i) first two minima and (ii) first two maxima on the screen.

Introduction: In this problem, a monochromatic beam of wavelength $\lambda = 630\,\text{nm} = 630 \times 10^{-9}\,\text{m}$ is incident on a narrow single slit of width $a = 0.2 \times 10^{-3}\,\text{m}$. The Fraunhofer diffraction pattern is projected onto a screen placed in the focal plane of a convex lens with focal length $f = 0.3\,\text{m}$. We are required to compute the linear distance on the screen between:

(i) The first two minima.

(ii) The first two maxima.

Solution:

(i) Distance between first two minima:

In single-slit diffraction, the angular positions of minima are given by:

$$a \sin\theta_m = m\lambda \quad \text{for } m = \pm 1, \pm 2, \ldots$$

For small angles, $\sin\theta_m \approx \tan\theta_m \approx \theta_m$, and the corresponding linear positions on the screen are:

$$y_m = f \tan\theta_m \approx f\theta_m = f\frac{m\lambda}{a}$$

The positions of the first two minima are:

$$y_1 = f\frac{\lambda}{a}, \quad y_2 = f\frac{2\lambda}{a}$$

Hence, the distance between the first two minima is:

$$\Delta y_{\text{minima}} = y_2 - y_1 = f\frac{\lambda}{a}$$

Substituting the given values:

$$\begin{aligned}
\Delta y_{\text{minima}} &= 0.3 \times \frac{630 \times 10^{-9}}{0.2 \times 10^{-3}} \\
&= 0.3 \times \frac{630 \times 10^{-9}}{0.2 \times 10^{-3}} \\
&= 0.3 \times 3.15 \times 10^{-3} \\
&= 0.945 \times 10^{-3}\,\text{m} = 0.945\,\text{mm}
\end{aligned}$$

(ii) Distance between first two maxima:

The "first two maxima" refers to the central maximum and the first secondary maximum. The central maximum extends from the first minimum on one side to the first minimum on the other side, while secondary maxima are located approximately halfway between consecutive minima.

The central maximum spans from $y = -f\frac{\lambda}{a}$ to $y = +f\frac{\lambda}{a}$, so its center is at $y = 0$.

The first secondary maximum is located approximately at:

$$y_{\text{1st secondary}} \approx \frac{y_1 + y_2}{2} = \frac{f\frac{\lambda}{a} + f\frac{2\lambda}{a}}{2} = f\frac{3\lambda}{2a}$$

Therefore, the distance between the central maximum and first secondary maximum is:

$$\Delta y_{\text{maxima}} = f\frac{3\lambda}{2a} - 0 = f\frac{3\lambda}{2a}$$

Substituting the given values:

$$\begin{aligned}\Delta y_{\text{maxima}} &= 0.3 \times \frac{3 \times 630 \times 10^{-9}}{2 \times 0.2 \times 10^{-3}} \\ &= 0.3 \times \frac{1890 \times 10^{-9}}{0.4 \times 10^{-3}} \\ &= 0.3 \times 4.725 \times 10^{-3} \\ &= 1.418 \times 10^{-3}\,\text{m} = 1.418\,\text{mm}\end{aligned}$$

Conclusion:

(i) The distance between the first two diffraction minima is 0.945 mm.

(ii) The distance between the first two maxima (central maximum and first secondary maximum) is 1.418 mm.

52 Explain the physical significance of resolving power of a grating with relevant mathematical expression.

Introduction: The resolving power of a diffraction grating is a measure of its ability to distinguish between two spectral lines of nearly equal wavelengths. In spectroscopy and optics, high resolving power is essential for separating closely spaced wavelengths in the emission or absorption spectra of substances.

Solution:

Definition and Mathematical Expression: The resolving power R of a diffraction grating is defined as:

$$R = \frac{\lambda}{\Delta\lambda}$$

where:

- λ is the mean wavelength of the two spectral lines.
- $\Delta\lambda$ is the smallest difference in wavelength that can be resolved at λ.

Theoretical Derivation: According to the Rayleigh criterion, two spectral lines are just resolved when the principal maximum of one coincides with the first minimum of the other.

For a diffraction grating with N slits, the condition for principal maxima is:

$$d \sin\theta = n\lambda$$

where d is the grating spacing, n is the order, and θ is the diffraction angle.

The angular width of a principal maximum is determined by the first minima on either side. The condition for the first minimum adjacent to the nth order maximum is:

$$Nd \sin\theta_{min} = n\lambda \pm \lambda$$

The angular separation between the principal maximum and its adjacent minimum is:

$$\Delta\theta = \frac{\lambda}{Nd\cos\theta}$$

For the Rayleigh criterion to be satisfied, two wavelengths λ and $\lambda + \Delta\lambda$ must have their maxima separated by this angular width:

$$\frac{d(\sin\theta)}{d\lambda}\Delta\lambda = \frac{\lambda}{Nd\cos\theta}$$

From the grating equation, differentiating:

$$\frac{d(\sin\theta)}{d\lambda} = \frac{n}{d\cos\theta}$$

Substituting this into the Rayleigh condition:

$$\frac{n}{d\cos\theta}\Delta\lambda = \frac{\lambda}{Nd\cos\theta}$$

Simplifying:

$$n\Delta\lambda = \frac{\lambda}{N}$$

Therefore, the resolving power is:

$$R = \frac{\lambda}{\Delta\lambda} = nN$$

Physical Significance: The expression $R = nN$ reveals several important physical aspects:

(i) **Order dependence**: Higher spectral orders (n) provide greater resolving power because the angular dispersion increases with order, spreading the spectrum more widely.

(ii) **Grating size effect**: More illuminated slits (N) increase resolving power because a larger grating produces sharper diffraction maxima, allowing finer discrimination between wavelengths.

(iii) **Finite aperture limitation**: The resolving power is fundamentally limited by the finite size of the grating aperture, which determines the sharpness of the diffraction pattern.

(iv) **Trade-off consideration**: While higher orders improve resolution, they also reduce the intensity of diffracted light, creating a practical trade-off in spectroscopic applications.

Practical Implications: The resolving power formula indicates that to achieve high spectral resolution, one must either:

- Use higher diffraction orders (limited by available light intensity)
- Employ gratings with more lines (larger physical size)
- Use both approaches in combination

This relationship explains why modern spectrometers use large gratings and why echelle spectrometers (which use very high orders) can achieve extremely high resolution.

Conclusion: The resolving power $R = \lambda/\Delta\lambda = nN$ quantifies a grating's capacity to distinguish closely spaced spectral lines. This formula reveals that resolution is fundamentally determined by the product of the diffraction order and the number of illuminated grating lines, reflecting the physical principle that spectral resolution depends on both angular dispersion and the sharpness of diffraction maxima.

53 Considering a plane transmission diffraction grating, where d is the distance between two consecutive ruled lines, m as the order number and θ as the angle of diffraction for normal incidence, calculate the angular dispersion $\frac{d\theta}{d\lambda}$ for an incident light of wavelength λ.

Introduction: We are given a plane transmission diffraction grating with line spacing d, used under normal incidence. The diffracted light forms maxima at angles θ satisfying the grating equation. We are to derive the expression for angular dispersion, defined as $\frac{d\theta}{d\lambda}$, which quantifies how much the diffraction angle θ changes with a small change in wavelength λ.

Angular dispersion is a fundamental property that determines how effectively a grating can separate different wavelengths spatially. The parameters involved are:

- d: grating spacing (distance between adjacent lines).
- m: order of diffraction.
- θ: angle of diffraction.
- λ: wavelength of incident light.

Solution: The diffraction condition for normal incidence is given by the grating equation:

$$d \sin\theta = m\lambda$$

Differentiating both sides with respect to λ, we get:

$$d \cos\theta \frac{d\theta}{d\lambda} = m$$

Solving for angular dispersion:

$$\frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta}$$

This is the required expression for angular dispersion.

Key insights from this expression:

(i) Angular dispersion increases with the order number m.

(ii) It also increases as $\cos\theta$ decreases, i.e., as θ increases.

(iii) Finer gratings (smaller d) produce higher angular dispersion.

Conclusion: The angular dispersion of a plane transmission diffraction grating under normal incidence is given by:

$$\frac{d\theta}{d\lambda} = \frac{m}{d\cos\theta}$$

It measures the rate at which the diffraction angle changes with wavelength and is crucial for the grating's spectral resolution capability.

54 Can D_1 and D_2 lines of sodium light ($\lambda_{D_1} = 5890$ Å and $\lambda_{D_2} = 5896$ Å) be resolved in second-order spectrum if the number of lines in the given grating is 450? Explain.

Introduction: The question asks whether the sodium doublet lines D_1 and D_2 with wavelengths $\lambda_{D_1} = 5890$ Å and $\lambda_{D_2} = 5896$ Å can be resolved in the second-order spectrum using a diffraction grating with $N = 450$ lines. This involves evaluating the resolving power R of the grating and comparing it with the required resolution to distinguish these two lines.

Solution:

The resolving power of a grating is given by:

$$R = \frac{\lambda}{\Delta\lambda} = nN$$

where:

- $n = 2$ is the order of diffraction.
- $N = 450$ is the total number of illuminated lines.
- $\Delta\lambda = \lambda_{D_2} - \lambda_{D_1} = 5896 - 5890 = 6\,\text{Å}$
- $\lambda = \frac{\lambda_{D_1}+\lambda_{D_2}}{2} = \frac{5890+5896}{2} = 5893\,\text{Å}$

Now compute the actual resolving power of the grating:

$$R_{\text{grating}} = nN = 2 \times 450 = 900$$

Compute the required resolving power to distinguish the two lines:

$$R_{\text{required}} = \frac{\lambda}{\Delta\lambda} = \frac{5893}{6} \approx 982.17$$

Since $R_{\text{grating}} = 900 < R_{\text{required}} = 982.17$, the grating does not have sufficient resolving power.

Conclusion:

The sodium D_1 and D_2 lines cannot be resolved in the second-order spectrum using a grating with only 450 lines, because the available resolving power (900) is less than the required resolving power (≈ 982).

55 Obtain an expression for the resolving power of a grating explaining the Rayleigh's criterion of resolution.

Introduction: The resolving power of a diffraction grating quantifies its ability to distinguish between two closely spaced spectral lines. According to Rayleigh's criterion, two spectral lines are just resolvable when the principal maximum of one coincides with the first minimum of the other. This criterion helps in deriving a quantitative expression for the resolving power of a grating.

Solution:

Consider a plane diffraction grating with N lines and grating spacing d. The condition for constructive interference (principal maxima) is:

$$d \sin\theta = n\lambda$$

where:

- d is the grating spacing.
- θ is the diffraction angle.
- n is the order of diffraction.
- λ is the wavelength of incident light.

Rayleigh's Criterion: Two wavelengths λ and $\lambda + \Delta\lambda$ are just resolvable when the principal maximum of one coincides with the first minimum of the other.

Derivation of Angular Width: For a grating with N slits, the condition for the first minimum adjacent to the principal maximum in the nth order is when the path difference between rays from the first and last slits differs by one wavelength from the condition for the principal maximum:

$$Nd \sin\theta_{min} = n\lambda \pm \lambda$$

The angular separation between the principal maximum and its adjacent first minimum is:

$$\Delta\theta = \theta_{min} - \theta = \frac{\lambda}{Nd\cos\theta}$$

This represents the angular half-width of the principal maximum.

Application of Rayleigh's Criterion: Consider two wavelengths λ and $\lambda + \Delta\lambda$. Their principal maxima occur at angles θ and $\theta + \Delta\theta_{sep}$ respectively, where:

$$d \sin\theta = n\lambda$$

$$d \sin(\theta + \Delta\theta_{sep}) = n(\lambda + \Delta\lambda)$$

Expanding the second equation for small $\Delta\theta_{sep}$:

$$d(\sin\theta + \cos\theta \cdot \Delta\theta_{sep}) = n\lambda + n\Delta\lambda$$

Subtracting the first equation:

$$d\cos\theta \cdot \Delta\theta_{sep} = n\Delta\lambda$$

Therefore:

$$\Delta\theta_{sep} = \frac{n\Delta\lambda}{d\cos\theta}$$

Resolution Condition: According to Rayleigh's criterion, the two wavelengths are just resolved when their angular separation equals the angular half-width of the diffraction maximum:

$$\Delta\theta_{sep} = \Delta\theta$$

Substituting the expressions:

$$\frac{n\Delta\lambda}{d\cos\theta} = \frac{\lambda}{Nd\cos\theta}$$

Simplifying:

$$n\Delta\lambda = \frac{\lambda}{N}$$

Therefore:

$$\Delta\lambda = \frac{\lambda}{nN}$$

Resolving Power: The resolving power is defined as:

$$R = \frac{\lambda}{\Delta\lambda} = \frac{\lambda}{\frac{\lambda}{nN}} = nN$$

Physical Interpretation: The expression $R = nN$ reveals that:

(i) Higher diffraction orders (n) increase resolving power due to greater angular dispersion.

(ii) More grating lines (N) produce sharper diffraction maxima, enabling finer wavelength discrimination.

(iii) The resolving power is independent of the grating spacing d and depends only on the total number of illuminated lines and the order used.

Conclusion: Using Rayleigh's criterion, the resolving power of a plane transmission diffraction grating is derived as:

$$R = nN$$

This expression shows that the resolving power increases with both the diffraction order n and the total number of slits N, indicating that higher orders and more lines yield finer spectral resolution.

56 Show that the areas of all the half-period zones are nearly the same. Find the radius of 1st half-period zone in a zone plate whose focal length is 50 cm and the wavelength of the incident light is 500 nm.

Introduction: A zone plate is an optical device made up of concentric rings known as Fresnel zones, which alternately block or transmit light to focus it through diffraction. These zones are defined such that the path difference between successive zones is half a wavelength. We aim to show that the areas of all half-period zones are nearly equal and compute the radius of the first half-period zone for a given focal length and wavelength.

Given:

- Focal length of zone plate, $f = 50\,\text{cm} = 0.5\,\text{m}$
- Wavelength of light, $\lambda = 500\,\text{nm} = 500 \times 10^{-9}\,\text{m}$

Solution:

Let r_n be the radius of the nth Fresnel zone. For a zone plate focusing light at a distance f from the plate, the radii of the zones are given by:

$$r_n = \sqrt{nf\lambda}$$

Hence, the area of the nth zone (between r_{n-1} and r_n) is:

$$A_n = \pi r_n^2 - \pi r_{n-1}^2 = \pi(r_n^2 - r_{n-1}^2)$$

Using the expression for r_n:

$$\begin{aligned} A_n &= \pi(nf\lambda - (n-1)f\lambda) \\ &= \pi f\lambda \end{aligned}$$

This shows that:

$$A_n = \pi f\lambda \quad \text{(constant for all } n\text{)}$$

Hence, the area of each half-period zone is approximately the same, independent of n.

Now, compute the radius of the 1st half-period zone ($n = 1$):

$$\begin{aligned} r_1 &= \sqrt{1 \cdot f \cdot \lambda} = \sqrt{0.5 \cdot 500 \times 10^{-9}} \\ &= \sqrt{2.5 \times 10^{-7}} \\ &= 5 \times 10^{-4}\,\text{m} = 0.5\,\text{mm} \end{aligned}$$

Conclusion:

- The areas of all half-period zones in a zone plate are approximately equal and given by $\pi f\lambda$.
- The radius of the 1st half-period zone for $f = 50\,\text{cm}$ and $\lambda = 500\,\text{nm}$ is 0.5 mm.

57 A plane transmission grating has 3000 lines in all, having width of 3 mm. What would be the angular separation in the first order spectrum of the two sodium lines of wavelengths 5890 Å and 5896 Å? Can they be seen distinctly?

Introduction: In this problem, a diffraction grating with 3000 lines and total width 3 mm is used to observe the sodium doublet at $\lambda_1 = 5890$ Å and $\lambda_2 = 5896$ Å. We are to calculate:

(i) The angular separation between the first-order diffraction angles of the two lines.

(ii) Whether the lines can be distinctly resolved using the grating.

Solution:

Given:

- Number of lines, $N = 3000$
- Grating width, $W = 3\,\text{mm} = 3 \times 10^{-3}\,\text{m}$
- Grating element, $d = \frac{W}{N} = \frac{3\times10^{-3}}{3000} = 10^{-6}\,\text{m}$
- Order of diffraction, $n = 1$
- $\lambda_1 = 5890\,\text{Å} = 5.890 \times 10^{-7}\,\text{m}$, $\lambda_2 = 5896\,\text{Å} = 5.896 \times 10^{-7}\,\text{m}$
- $\Delta\lambda = \lambda_2 - \lambda_1 = 6 \times 10^{-10}\,\text{m}$

(i) Angular separation:

Method 1: Using angular dispersion formula The angular dispersion of a grating is given by:

$$\frac{d\theta}{d\lambda} = \frac{n}{d\cos\theta}$$

For the mean wavelength $\bar{\lambda} = \frac{\lambda_1+\lambda_2}{2} = 5893\,\text{Å}$:

$$\sin\theta = \frac{n\bar{\lambda}}{d} = \frac{1 \times 5.893 \times 10^{-7}}{10^{-6}} = 0.5893$$

$$\theta = \sin^{-1}(0.5893) \approx 36.04^\circ$$

$$\cos\theta = \cos(36.04^\circ) \approx 0.808$$

Therefore:

$$\frac{d\theta}{d\lambda} = \frac{1}{10^{-6} \times 0.808} = 1.238 \times 10^{6}\,\text{rad/m}$$

Angular separation:

$$\Delta\theta = \frac{d\theta}{d\lambda} \times \Delta\lambda = 1.238 \times 10^{6} \times 6 \times 10^{-10} = 7.43 \times 10^{-4}\,\text{rad}$$

Converting to degrees:

$$\Delta\theta = 7.43 \times 10^{-4} \times \frac{180}{\pi} \approx 0.043^\circ$$

Method 2: Direct calculation (verification) Using the grating equation for normal incidence:

$$d \sin\theta = n\lambda$$

For first-order ($n = 1$):

$$\sin\theta_1 = \frac{5.890 \times 10^{-7}}{10^{-6}} = 0.589$$
$$\sin\theta_2 = \frac{5.896 \times 10^{-7}}{10^{-6}} = 0.5896$$

Computing the angles:

$$\theta_1 = \sin^{-1}(0.589) \approx 36.02^\circ$$
$$\theta_2 = \sin^{-1}(0.5896) \approx 36.06^\circ$$

Angular separation:

$$\Delta\theta = \theta_2 - \theta_1 \approx 0.04^\circ$$

Both methods give consistent results.

(ii) Resolving power:

Required resolving power:

$$R_{\text{required}} = \frac{\bar{\lambda}}{\Delta\lambda} = \frac{5893 \times 10^{-10}}{6 \times 10^{-10}} \approx 982.17$$

Grating resolving power:

$$R_{\text{grating}} = nN = 1 \times 3000 = 3000$$

Since $R_{\text{grating}} = 3000 > R_{\text{required}} = 982.17$, the grating can resolve the lines with a safety margin of about 3.

Conclusion:

- The angular separation between the first-order diffraction maxima of the sodium D-lines is approximately 0.043° or 7.43×10^{-4} radians.
- The lines can be distinctly resolved, as the resolving power of the grating (3000) significantly exceeds the required value (≈ 982).

58 Discuss the intensity distribution in Fraunhofer diffraction pattern due to a single slit. Obtain conditions for maxima and minima of the intensity distribution. Show that the intensity of the first maxima is about 4.95% of that of the principal maxima.

Introduction: In Fraunhofer diffraction due to a single slit, a plane wavefront is incident normally on a slit of finite width. The light diffracts and forms a pattern of alternating bright and dark fringes on a screen placed at the focal plane of a converging lens. We aim to derive the mathematical expression for the intensity distribution, obtain the conditions for minima and maxima, and calculate the relative intensity of the first secondary maximum.

Solution:

Derivation of Intensity Distribution: Let:

- a be the width of the slit,
- λ be the wavelength of light,
- θ be the angle of diffraction.

Consider the slit divided into many narrow strips of width dy. Each strip acts as a source of secondary wavelets. The path difference between rays from a strip at distance y from the center and the central ray is:

$$\delta = y \sin\theta$$

The corresponding phase difference is:

$$\phi = \frac{2\pi}{\lambda} y \sin\theta$$

The amplitude contribution from each strip is $dE = \frac{E_0}{a} dy$, where E_0 is the total amplitude when all rays are in phase.

The resultant amplitude is:

$$E = \int_{-a/2}^{a/2} \frac{E_0}{a} e^{i\phi} dy = \frac{E_0}{a} \int_{-a/2}^{a/2} e^{i\frac{2\pi y \sin\theta}{\lambda}} dy$$

Let $k = \frac{2\pi \sin\theta}{\lambda}$, then:

$$E = \frac{E_0}{a} \int_{-a/2}^{a/2} e^{iky} dy = \frac{E_0}{a} \left[\frac{e^{iky}}{ik} \right]_{-a/2}^{a/2}$$

$$E = \frac{E_0}{a} \cdot \frac{1}{ik} \left(e^{ika/2} - e^{-ika/2} \right) = \frac{E_0}{a} \cdot \frac{2\sin(ka/2)}{k}$$

Substituting $k = \frac{2\pi \sin\theta}{\lambda}$:

$$E = E_0 \cdot \frac{\sin\left(\frac{\pi a \sin\theta}{\lambda}\right)}{\frac{\pi a \sin\theta}{\lambda}}$$

Let $\beta = \frac{\pi a \sin\theta}{\lambda}$, then:

$$E = E_0 \frac{\sin\beta}{\beta}$$

The intensity is proportional to $|E|^2$:

$$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2$$

where I_0 is the intensity at $\theta = 0$ (central maximum).

Conditions for Minima: Minima occur when $\sin\beta = 0$ (excluding $\beta = 0$), i.e.,

$$\beta = \pm m\pi \quad (m = 1, 2, 3, \ldots)$$

Thus:

$$\frac{\pi a \sin\theta}{\lambda} = m\pi \Rightarrow a \sin\theta = m\lambda$$

So, the angular positions of minima are:

$$\sin\theta_m = \frac{m\lambda}{a} \quad (m = \pm1, \pm2, \pm3, \ldots)$$

Conditions for Maxima: The central maximum occurs at $\theta = 0$ where $\beta = 0$.

For secondary maxima, we differentiate the intensity function:

$$\frac{d}{d\beta}\left(\frac{\sin\beta}{\beta}\right)^2 = 0$$

This leads to:

$$\frac{d}{d\beta}\left(\frac{\sin\beta}{\beta}\right) = 0$$

$$\frac{\beta\cos\beta - \sin\beta}{\beta^2} = 0$$

This gives the condition:

$$\tan\beta = \beta$$

This transcendental equation has solutions at $\beta \approx 1.4303\pi, 2.4590\pi, 3.4707\pi, \ldots$ for the first, second, third secondary maxima, respectively.

Relative Intensity of First Secondary Maximum: The first secondary maximum occurs at $\beta_1 \approx 1.4303\pi = 4.493$.

The intensity at this point is:

$$I_1 = I_0 \left(\frac{\sin(4.493)}{4.493}\right)^2$$

Since $\sin(4.493) \approx -0.975$ (taking absolute value for intensity):

$$I_1 = I_0 \left(\frac{0.975}{4.493}\right)^2 = I_0(0.217)^2 \approx 0.0471 I_0$$

More precisely:

$$I_1 \approx 0.0472 I_0 \approx 4.72\% \text{ of } I_0$$

This is very close to the stated 4.95

Physical Interpretation: The rapid decrease in intensity of secondary maxima occurs because destructive interference becomes increasingly dominant as we move away from the central maximum. The $(sin\beta/\beta)^2$ function ensures that most of the diffracted light is concentrated in the central maximum.

Conclusion:

- The intensity distribution in Fraunhofer diffraction from a single slit is given by:

$$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta} \right)^2, \quad \beta = \frac{\pi a \sin\theta}{\lambda}$$

- Minima occur at $a \sin\theta = m\lambda$, $m = \pm 1, \pm 2, \ldots$
- Secondary maxima are determined by the condition $\tan\beta = \beta$
- The intensity of the first secondary maximum is approximately 4.72% of that of the central (principal) maximum.

59 Show that the phenomenon of Fraunhofer diffraction at two vertical slits is modulation of two terms viz. double slit interference and single slit diffraction. Obtain the condition for positions of maxima and minima.

Introduction: When monochromatic light undergoes Fraunhofer diffraction at two vertical slits of finite width, the resulting intensity pattern exhibits a unique characteristic: the sharp interference fringes from double-slit interference are modulated by the broader envelope of single-slit diffraction. This occurs because each slit acts as both a diffracting aperture and an interference source.

Solution: Let:

- a be the width of each slit,
- d be the distance between the centers of the two slits,
- λ be the wavelength of incident light,
- θ be the angle of observation from the central axis.

Step 1: Single-Slit Diffraction Pattern For a single slit of width a, the intensity distribution is:

$$I_{\text{single}}(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2, \quad \text{where } \beta = \frac{\pi a \sin\theta}{\lambda}$$

This pattern has a central maximum and subsidiary maxima with decreasing intensity.

Step 2: Double-Slit Interference Pattern For two infinitesimally narrow slits separated by distance d, the interference pattern is:

$$I_{\text{interference}}(\theta) = I_0 \cos^2\delta, \quad \text{where } \delta = \frac{\pi d \sin\theta}{\lambda}$$

This produces equally spaced fringes with equal intensity.

Step 3: Physical Origin of Modulation When slits have finite width, each slit contributes to both diffraction and interference simultaneously:

- Each slit diffracts light according to its individual diffraction pattern
- The two slits interfere with each other based on their separation
- The resultant is the interference pattern modulated by the diffraction envelope

Step 4: Combined Intensity Pattern The total intensity is the product of the two individual effects:

$$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2 \cos^2\delta$$

where:

$$\beta = \frac{\pi a \sin\theta}{\lambda}, \quad \delta = \frac{\pi d \sin\theta}{\lambda}$$

This demonstrates that the interference fringes are modulated by the single-slit diffraction envelope. The $\cos^2\delta$ term creates the interference fringes, while the $\left(\frac{\sin\beta}{\beta}\right)^2$ term acts as a modulating envelope.

Conditions for Maxima: Primary maxima occur when both terms are maximum:

- Interference maxima: $\cos^2\delta = 1 \Rightarrow \delta = m\pi$
- This gives: $d\sin\theta = m\lambda$ where $m = 0, \pm 1, \pm 2, \ldots$

However, the actual intensity depends on the diffraction envelope at these positions.

Conditions for Minima: Minima occur when either term becomes zero:

(i) Interference minima:

$$\cos^2\delta = 0 \Rightarrow \delta = (2n+1)\frac{\pi}{2}$$

$$\Rightarrow d\sin\theta = (2n+1)\frac{\lambda}{2} \quad \text{where } n = 0, \pm 1, \pm 2, \ldots$$

(ii) Diffraction minima:

$$\frac{\sin\beta}{\beta} = 0 \Rightarrow \sin\beta = 0 \text{ and } \beta \neq 0$$

$$\Rightarrow \beta = p\pi \text{ where } p = \pm 1, \pm 2, \pm 3, \ldots$$

$$\Rightarrow a\sin\theta = p\lambda$$

Missing Orders: Some interference maxima may be missing when they coincide with diffraction minima, i.e., when:

$$\frac{d}{a} = \frac{p}{m}$$

Conclusion: The Fraunhofer diffraction pattern from two finite-width slits demonstrates the modulation principle where:

- The intensity pattern: $I(\theta) = I_0\left(\frac{\sin\beta}{\beta}\right)^2\cos^2\delta$
- Represents interference fringes modulated by a diffraction envelope
- Maxima occur at $d\sin\theta = m\lambda$ (subject to diffraction envelope)
- Minima arise from either interference: $d\sin\theta = (2n+1)\frac{\lambda}{2}$ or diffraction: $a\sin\theta = p\lambda$
- The phenomenon illustrates the superposition of wave effects in optics

60 Discuss the phenomenon of Fraunhofer diffraction at a single slit and show that the intensities of successive maxima are nearly in the ratio $1 : \frac{4}{9\pi^2} : \frac{4}{25\pi^2} : \frac{4}{49\pi^2}$.

Introduction: Fraunhofer diffraction at a single slit occurs when parallel monochromatic light passes through a narrow aperture and is observed at a large distance (far-field approximation) or at the focal plane of a converging lens. This phenomenon demonstrates the wave nature of light and produces a characteristic intensity pattern with a dominant central maximum and progressively weaker secondary maxima.

Physical Description of the Phenomenon: When plane waves encounter a single slit of width a, each point within the slit acts as a secondary source of spherical wavelets (Huygens' principle). The interference of these wavelets produces the observed diffraction pattern. The intensity varies with angle due to the path difference between rays from different parts of the slit.

Mathematical Analysis: Let:

- a be the width of the slit,
- λ be the wavelength of light,
- θ be the angle of diffraction from the normal.

The intensity distribution for single-slit Fraunhofer diffraction is:

$$I(\theta) = I_0 \left(\frac{\sin\beta}{\beta}\right)^2, \quad \text{where } \beta = \frac{\pi a \sin\theta}{\lambda}$$

Condition for Minima: Minima occur when the numerator is zero but the denominator is non-zero:

$$\sin\beta = 0 \text{ and } \beta \neq 0$$
$$\Rightarrow \beta = m\pi \quad (m = \pm1, \pm2, \pm3, \ldots)$$
$$\Rightarrow a\sin\theta = m\lambda$$

Condition for Secondary Maxima: Secondary maxima occur where $\frac{d}{d\beta}\left(\frac{\sin\beta}{\beta}\right)^2 = 0$, which leads to:

$$\frac{d}{d\beta}\left(\frac{\sin\beta}{\beta}\right) = 0$$
$$\Rightarrow \frac{\beta\cos\beta - \sin\beta}{\beta^2} = 0$$
$$\Rightarrow \tan\beta = \beta$$

This transcendental equation has solutions approximately at:

$$\beta_m \approx \left(m + \frac{1}{2}\right)\pi \quad (m = 1, 2, 3, \ldots)$$

The approximation becomes more accurate for larger values of m.

Calculation of Intensity Ratios: For the central maximum ($\beta = 0$):

$$I_0 = I_0 \lim_{\beta\to0}\left(\frac{\sin\beta}{\beta}\right)^2 = I_0$$

For secondary maxima at $\beta_m \approx \left(m + \frac{1}{2}\right)\pi$:

$$I_m = I_0 \left(\frac{\sin\left[\left(m + \frac{1}{2}\right)\pi\right]}{\left(m + \frac{1}{2}\right)\pi} \right)^2$$

Since $\sin\left[\left(m + \frac{1}{2}\right)\pi\right] = (-1)^m$, we have:

$$I_m = I_0 \left(\frac{(-1)^m}{\left(m + \frac{1}{2}\right)\pi} \right)^2 = I_0 \left(\frac{1}{\left(m + \frac{1}{2}\right)\pi} \right)^2$$

Computing for successive maxima:

(i) Central maximum: $I_0 = I_0$

(ii) First secondary maximum ($m = 1$):

$$I_1 = I_0 \left(\frac{1}{\frac{3\pi}{2}} \right)^2 = I_0 \left(\frac{2}{3\pi} \right)^2 = I_0 \cdot \frac{4}{9\pi^2}$$

(iii) Second secondary maximum ($m = 2$):

$$I_2 = I_0 \left(\frac{1}{\frac{5\pi}{2}} \right)^2 = I_0 \left(\frac{2}{5\pi} \right)^2 = I_0 \cdot \frac{4}{25\pi^2}$$

(iv) Third secondary maximum ($m = 3$):

$$I_3 = I_0 \left(\frac{1}{\frac{7\pi}{2}} \right)^2 = I_0 \left(\frac{2}{7\pi} \right)^2 = I_0 \cdot \frac{4}{49\pi^2}$$

General Pattern: The intensity of the m-th secondary maximum is:

$$I_m = I_0 \cdot \frac{4}{(2m+1)^2\pi^2}$$

Intensity Ratio: Therefore, the ratio of successive maxima is:

$$I_0 : I_1 : I_2 : I_3 = 1 : \frac{4}{9\pi^2} : \frac{4}{25\pi^2} : \frac{4}{49\pi^2}$$

Numerical Values:

$$1 : 0.045 : 0.016 : 0.0083$$

Conclusion: The Fraunhofer diffraction pattern at a single slit exhibits a characteristic intensity distribution with a dominant central maximum and rapidly decreasing secondary maxima. The intensity ratios follow the pattern $1 : \frac{4}{9\pi^2} : \frac{4}{25\pi^2} : \frac{4}{49\pi^2}$, demonstrating the wave nature of light and the interference effects within the aperture. This rapid decrease in intensity explains why only the central maximum and first few secondary maxima are typically observable in practice.