UPSC PHYSICS PYQ SOLUTION

Waves and Optics - Part 7

Contents

69 A pulse of $\lambda_0 = 600$ **nm and** $\Delta\lambda = 10$ **nm propagates through a fibre which has a material dispersion coefficient of 50 ps per km per nm at 600 nm. Calculate the pulse broadening in traversing a 10 km length of the fibre. If the pulse width at the input of the fibre is 12 ns, what will be the pulse width at the output of the fibre?** **19**

70 Calculate the minimum thickness of a quartz plate which would behave as a quarter-wave plate for wavelength of light, $\lambda = 6000$ **Å. The refractive indices for ordinary and extraordinary rays are** $\mu_o = 1.544$ **and** $\mu_e = 1.553$**.** **20**

61 An unpolarized light beam of intensity $1000\ \mathrm{W/m^2}$ is incident on an ideal linear polarizer with its transmission axis parallel to vertical direction. Describe an experiment to reduce the intensity of light beam to $500\ \mathrm{W/m^2}$.

Introduction: The problem involves designing an experiment to reduce the intensity of an unpolarized light beam from $1000\ \mathrm{W/m^2}$ to $500\ \mathrm{W/m^2}$ using polarizers. We need to consider the behavior of unpolarized light when it interacts with polarizing elements.

Theory: When unpolarized light of intensity I_0 passes through an ideal linear polarizer, the transmitted intensity is:

$$I_1 = \frac{I_0}{2}$$

When polarized light of intensity I_1 passes through a second polarizer making an angle θ with the first polarizer's transmission axis, the transmitted intensity follows Malus's law:

$$I_2 = I_1 \cos^2(\theta)$$

Experimental Setup - Method 1 (Single Polarizer):

Equipment needed:

- Unpolarized light source (intensity $1000\ \mathrm{W/m^2}$)
- One ideal linear polarizer
- Light intensity meter/photodetector
- Optical bench or stable mounting system

Procedure:

1. Mount the unpolarized light source on an optical bench.
2. Place the linear polarizer with its transmission axis vertical in the beam path.
3. Position the intensity meter to measure the transmitted light.
4. Record the intensity measurement.

Result: The transmitted intensity will be:

$$I_1 = \frac{I_0}{2} = \frac{1000}{2} = 500\ \mathrm{W/m^2}$$

Experimental Setup - Method 2 (Two Polarizers):

Alternatively, we can achieve the same result using two polarizers:

Equipment needed:

- Same as Method 1, plus one additional polarizer
- Protractor or angular measurement device

Procedure:

1. Set up the first polarizer as in Method 1.
2. Place a second polarizer after the first, initially aligned parallel to the first.
3. Rotate the second polarizer until the desired intensity is achieved.
4. Measure the angle between the two polarizers.

For this method, if we want the final intensity to be 500 W/m^2:

- After first polarizer: $I_1 = 500 \text{ W/m}^2$
- After second polarizer: $I_2 = I_1 \cos^2(\theta) = 500 \text{ W/m}^2$

This requires $\cos^2(\theta) = 1$, so $\theta = 0°$ (polarizers aligned), which is equivalent to Method 1.

To demonstrate the versatility of this setup, we could also achieve other intensities. For example, with $I_0 = 1000 \text{ W/m}^2$:

- After first polarizer: $I_1 = 500 \text{ W/m}^2$
- To get 250 W/m^2: second polarizer at $\theta = 45°$
- To get 125 W/m^2: second polarizer at $\theta = 60°$

Experimental Considerations:

- Ensure polarizers are truly ideal (minimal absorption losses).
- Verify that the incident light is completely unpolarized.
- Account for any reflection losses at polarizer surfaces.
- Maintain stable alignment during measurements.

Conclusion: The simplest experimental approach uses a single linear polarizer, which naturally reduces unpolarized light intensity by exactly half, achieving the target intensity of 500 W/m^2. This demonstrates the fundamental principle that polarizers transmit only one component of unpolarized light, effectively filtering out half the incident intensity.

A/P

62 What should be the refractive index of cladding of an optical fibre with numerical aperture 0.5 with refractive index of core as 1.5?

Introduction: The problem requires determining the refractive index of the cladding (n_2) of an optical fibre, given the numerical aperture (NA) and the refractive index of the core (n_1). The numerical aperture is a function of both refractive indices, and it quantifies the fibre's ability to collect light. Given: $\text{NA} = 0.5$, $n_1 = 1.5$.

Solution:

The numerical aperture of an optical fibre is defined as:

$$\text{NA} = \sqrt{n_1^2 - n_2^2}$$

We are given:

$$\text{NA} = 0.5$$
$$n_1 = 1.5$$

Substituting into the NA equation:

$$0.5 = \sqrt{1.5^2 - n_2^2}$$

Squaring both sides:

$$0.25 = 2.25 - n_2^2$$

Rearranging:

$$n_2^2 = 2.25 - 0.25 = 2.00$$

Taking the square root:

$$n_2 = \sqrt{2.00} = 1.414$$

Conclusion: The refractive index of the cladding should be approximately 1.414 to achieve a numerical aperture of 0.5 with a core refractive index of 1.5.

A/P

63 A laser beam of 1 micrometer wavelength with 3 megawatts power of beam diameter 10 mm is focussed by a lens of focal length 50 mm. Evaluate the electric field associated with the light beam at the focal point. (Dielectric permittivity of free space, $\epsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$)

Introduction: The problem requires computing the electric field magnitude at the focal point of a lens focusing a laser beam. The given parameters are:

- Wavelength of light, $\lambda = 1\ \mu\mathrm{m} = 1 \times 10^{-6}$ m
- Power of beam, $P = 3$ MW $= 3 \times 10^6$ W
- Diameter of the beam at the lens, $D = 10$ mm $= 0.01$ m
- Radius of the beam at the lens, $w_{in} = D/2 = 0.005$ m
- Focal length of the lens, $f = 50$ mm $= 0.05$ m
- Permittivity of free space, $\epsilon_0 = 8.8542 \times 10^{-12}\ \mathrm{C^2/N \cdot m^2}$

We aim to compute the electric field amplitude E_0 at the focal spot.

Solution:

For a Gaussian beam focused by a lens, the beam waist (radius) at the focal point, w_0, is given by:

$$w_0 = \frac{\lambda f}{\pi w_{in}}$$

Here, w_{in} is the radius of the beam incident on the lens. Given the diameter $D = 10$ mm, the radius is $w_{in} = D/2 = 5$ mm $= 0.005$ m.

Substituting the given values:

$$w_0 = \frac{(1 \times 10^{-6}\ \mathrm{m}) \times (0.05\ \mathrm{m})}{\pi \times (0.005\ \mathrm{m})} = \frac{5 \times 10^{-8}}{\pi \times 0.005} = \frac{1 \times 10^{-7}}{\pi} \approx 3.183 \times 10^{-8}\ \mathrm{m}$$

The area of the focused spot is:

$$A = \pi w_0^2 = \pi (3.183 \times 10^{-8})^2 \approx \pi \times 1.013 \times 10^{-15} \approx 3.183 \times 10^{-15}\ \mathrm{m}^2$$

The intensity (power per unit area) at the focal point is:

$$I = \frac{P}{A} = \frac{3 \times 10^6\ \mathrm{W}}{3.183 \times 10^{-15}\ \mathrm{m}^2} \approx 9.425 \times 10^{20}\ \mathrm{W/m^2}$$

The relationship between the intensity I and the amplitude of the electric field E_0 is given by:

$$I = \frac{1}{2} c \epsilon_0 E_0^2$$

where c is the speed of light in vacuum ($c \approx 3 \times 10^8$ m/s).

Solving for E_0:

$$E_0 = \sqrt{\frac{2I}{c\epsilon_0}} = \sqrt{\frac{2 \times (9.425 \times 10^{20})}{(3 \times 10^8)(8.8542 \times 10^{-12})}}$$

$$E_0 = \sqrt{\frac{1.885 \times 10^{21}}{2.65626 \times 10^{-3}}} = \sqrt{7.096 \times 10^{23}} \approx 8.42 \times 10^{11} \text{ V/m}$$

Conclusion: The electric field amplitude at the focal point of the focused laser beam is approximately 8.42×10^{11} V/m.

A/P

64 A plane wave has the following expression for its electric field: $\vec{E} = \hat{x}E_{0x}\cos(\omega t - kz + \alpha) + \hat{y}E_{0y}\cos(\omega t - kz + \beta)$. If the phase difference is defined as $\delta = \beta - \alpha$, under what conditions do we achieve elliptic polarization? What are the conditions for circular polarization?

Introduction: The electric field of a plane wave is given with components in the x and y directions, each having distinct amplitudes and phase shifts. The polarization of light is determined by the relative amplitudes and phases of these orthogonal components. The problem requires identifying the conditions under which the wave exhibits elliptic or circular polarization.

Solution: The general form of the electric field is:

$$\vec{E}(z,t) = \hat{x}E_{0x}\cos(\omega t - kz + \alpha) + \hat{y}E_{0y}\cos(\omega t - kz + \beta)$$

Let us define the phase difference:

$$\delta = \beta - \alpha$$

To analyze polarization, we examine the trajectory traced by the tip of the electric field vector $\vec{E}$ at a fixed point in space (e.g., $z = 0$). Setting $z = 0$ and defining $\phi = \omega t + \alpha$:

$$E_x = E_{0x}\cos(\phi)$$
$$E_y = E_{0y}\cos(\phi + \delta)$$

Using the trigonometric identity $\cos(\phi + \delta) = \cos(\phi)\cos(\delta) - \sin(\phi)\sin(\delta)$:

$$E_y = E_{0y}[\cos(\phi)\cos(\delta) - \sin(\phi)\sin(\delta)]$$

From the first equation: $\cos(\phi) = \frac{E_x}{E_{0x}}$ and $\sin(\phi) = \pm\sqrt{1 - \frac{E_x^2}{E_{0x}^2}}$

Substituting and eliminating the time-dependent terms leads to the general equation:

$$\frac{E_x^2}{E_{0x}^2} + \frac{E_y^2}{E_{0y}^2} - \frac{2E_xE_y}{E_{0x}E_{0y}}\cos(\delta) = \sin^2(\delta)$$

This is the equation of an ellipse in the E_x-E_y plane for general values of E_{0x}, E_{0y}, and δ.

Classification of Polarization States:

(i) **Linear Polarization:** Occurs when $\delta = n\pi$ (where n is any integer):

- When $\delta = 0, 2\pi, 4\pi, ...$: $E_y = \pm\frac{E_{0y}}{E_{0x}}E_x$ (positive slope)
- When $\delta = \pi, 3\pi, 5\pi, ...$: $E_y = \mp\frac{E_{0y}}{E_{0x}}E_x$ (negative slope)

The electric field vector oscillates along a straight line.

(ii) **Elliptic Polarization:** This is the general case and occurs when:

- $E_{0x} \neq 0$ and $E_{0y} \neq 0$, and

- $\delta \neq n\pi$ for any integer n

The electric field vector traces an ellipse in the xy-plane.

(iii) **Circular Polarization:** This is a special case of elliptical polarization. The conditions are:

- $E_{0x} = E_{0y}$, and
- $\delta = \pm\frac{\pi}{2} + 2n\pi$ (where n is any integer)

When $\delta = +\frac{\pi}{2}$: Left-hand circular polarization When $\delta = -\frac{\pi}{2}$: Right-hand circular polarization

For circular polarization, the ellipse equation reduces to:

$$E_x^2 + E_y^2 = E_0^2$$

which is a circle of radius $E_0 = E_{0x} = E_{0y}$.

Conclusion:

- The wave is elliptically polarized when both E_{0x} and E_{0y} are nonzero and the phase difference δ is not a multiple of π.
- The wave is circularly polarized when $E_{0x} = E_{0y}$ and $\delta = \pm\frac{\pi}{2}$ (plus any multiple of 2π).
- Linear polarization occurs as a special case when $\delta = n\pi$.

A/P

65 For calcite, the refractive indices of ordinary and extraordinary rays are 1.65836 and 1.48641 at $\lambda_0 = 5893$ Å respectively. A left circularly polarized beam of this wavelength is incident normally on such crystal of thickness 0.005141 mm having its optic axis cut parallel to the surface. What will be the state of polarization of the emergent beam?

Introduction: A left circularly polarized (LCP) beam is incident on a birefringent calcite crystal. We need to determine the polarization of the emergent beam by calculating the phase difference introduced between the ordinary and extraordinary rays.

The given parameters are:

- Refractive index for ordinary ray, $n_o = 1.65836$
- Refractive index for extraordinary ray, $n_e = 1.48641$
- Wavelength in vacuum, $\lambda_0 = 5893$ Å $= 5.893 \times 10^{-7}$ m
- Thickness of the crystal, $d = 0.005141$ mm $= 5.141 \times 10^{-6}$ m

Solution:

Step 1: Calculate the phase difference ($\Delta\phi$) When a light beam travels through a birefringent material of thickness d, a phase difference is introduced between its ordinary and extraordinary components. This phase difference is given by the formula:

$$\Delta\phi = \frac{2\pi}{\lambda_0}(n_o - n_e)d$$

First, let's calculate the difference in refractive indices:

$$n_o - n_e = 1.65836 - 1.48641 = 0.17195$$

Now, substitute the values into the phase difference formula:

$$\Delta\phi = \frac{2\pi}{5.893 \times 10^{-7} \text{ m}}(0.17195)(5.141 \times 10^{-6} \text{ m})$$

$$\Delta\phi = 2\pi \times \frac{0.17195 \times 5.141 \times 10^{-6}}{5.893 \times 10^{-7}}$$

$$\Delta\phi = 2\pi \times \frac{0.88399 \times 10^{-6}}{5.893 \times 10^{-7}} = 2\pi \times (0.150007 \times 10^{1})$$

$$\Delta\phi \approx 2\pi \times 1.5 = 3\pi \text{ radians}$$

Step 2: Analyze the effect on polarization The phase difference $\Delta\phi = 3\pi$ radians indicates that the crystal is acting as a **half-wave plate** for the given wavelength. A half-wave plate introduces a phase shift of π (or any odd multiple of π) between the two orthogonal components of polarization.

The effect of a half-wave plate on circularly polarized light is to reverse its handedness:

- Left circularly polarized light becomes right circularly polarized.
- Right circularly polarized light becomes left circularly polarized.

Since the incident beam is left circularly polarized, it will emerge from the crystal as right circularly polarized.

Conclusion: The emergent beam will be **right circularly polarized**. The crystal introduces a phase difference of 3π, acting as a half-wave plate and reversing the handedness of the incident circularly polarized light.

A/P

66 Bring out the essential differences between the physical principles of spontaneous and stimulated emission of radiation. Why is it difficult to get efficient lasing action in case of an ideal two-level material system? Can you propose a scheme to enhance efficiency? Discuss.

Introduction: The question seeks to clarify the fundamental differences between spontaneous and stimulated emission processes, explain the inherent inefficiency of lasing in a simple two-level system, and propose a more effective scheme for achieving laser action.

Differences between Spontaneous and Stimulated Emission:

(i) **Spontaneous Emission**: This is the natural emission of a photon from an atom or molecule in an excited energy state.

- **Trigger**: Occurs randomly, without any external electromagnetic field to trigger the transition.
- **Coherence**: The emitted photons are **incoherent**. Their phase, direction, and polarization are random.
- **Rate**: The rate of emission is proportional to the number of atoms in the excited state (N_2) and is governed by the Einstein A coefficient (A_{21}). Rate $= A_{21}N_2$.
- **Role**: It is the primary mechanism behind the light from conventional sources like light bulbs and stars. It also represents a loss mechanism in lasers.

(ii) **Stimulated Emission**: This process occurs when an incoming photon interacts with an already excited atom, causing it to de-excite and emit a second photon.

- **Trigger**: Requires an external photon of a specific energy ($h\nu = E_2 - E_1$) to initiate the emission.
- **Coherence**: The emitted photon is a perfect copy of the stimulating photon. It is **coherent**, having the same frequency, phase, direction, and polarization.
- **Rate**: The rate is proportional to the number of excited atoms (N_2) and the energy density of the stimulating radiation field ($\rho(\nu)$). It is governed by the Einstein B coefficient (B_{21}). Rate $= B_{21}N_2\rho(\nu)$.
- **Role**: This is the fundamental principle of light amplification in lasers (Light Amplification by Stimulated Emission of Radiation).

Lasing Difficulty in a Two-Level System:

Achieving efficient lasing in a two-level system (E_1, E_2) is practically impossible due to the following reasons:

- **Competing Processes**: The same photons used for pumping (exciting atoms from $E_1 \to E_2$) have the exact energy required to stimulate emission ($E_2 \to E_1$). Therefore, absorption and stimulated emission are competing processes that occur at the same frequency.

- **Lack of Population Inversion**: For light amplification (gain), the rate of stimulated emission must exceed the rate of absorption. This requires a **population inversion**, where more atoms are in the excited state than the lower state ($N_2 > N_1$). In a two-level system, even with extremely intense pumping, the best one can achieve is equal populations ($N_2 \approx N_1$). At this point, the rates of absorption and stimulated emission become equal, and the material becomes transparent to the radiation but provides no net gain. It is thermodynamically impossible to achieve $N_2 > N_1$ with optical pumping in a simple two-level system.

Proposed Scheme for Enhanced Efficiency: Multi-Level Systems:

To overcome this fundamental limitation, practical lasers use materials with three or four energy levels. The four-level system is generally more efficient.

The Four-Level Laser System (e.g., Nd:YAG laser): This scheme uses four relevant energy levels:

1. E_0: The ground state.
2. E_3: A short-lived, broad "pump band".
3. E_2: A long-lived **metastable state** (the upper laser level).
4. E_1: A short-lived lower laser level.

The process is as follows:

- **Pumping**: An external source excites atoms from the ground state E_0 to the pump band E_3.
- **Fast Decay (1)**: Atoms in E_3 very quickly and non-radiatively (e.g., via vibrations) decay to the metastable upper laser level E_2. Because this decay is fast, atoms accumulate in E_2.
- **Lasing Transition**: A population inversion is achieved between E_2 and E_1 ($N_2 > N_1$). Stimulated emission occurs, causing atoms to transition from $E_2 \to E_1$, releasing coherent laser photons.
- **Fast Decay (2)**: Atoms in the lower laser level E_1 rapidly decay back to the ground state E_0.

Discussion of Enhanced Efficiency: The four-level scheme is highly efficient because the lower laser level (E_1) is not the ground state (E_0) and is kept nearly empty due to its rapid decay. Therefore, only a small number of atoms need to be pumped to the upper level (E_2) to achieve population inversion ($N_2 > N_1 \approx 0$). This drastically reduces the pumping energy required (the laser threshold) compared to a three-level system, where the lower laser level is the ground state and more than half of all atoms must be pumped to achieve inversion.

Conclusion: Spontaneous and stimulated emission differ fundamentally in their triggers and the coherence of the resulting radiation. A two-level system is unsuitable for efficient lasing because pumping and emission occur at the same frequency, making population inversion impossible to achieve. The introduction of additional energy levels, particularly in a four-level scheme, separates the pump and lasing transitions and utilizes a rapidly decaying lower laser level, allowing for efficient and low-threshold laser operation.

67 Show with proper mathematical analysis that the ratio of Einstein's A and B coefficients depends upon the energy separation between the two energy levels participating in the optical transitions. What is the physical significance of A coefficient? Justify the statement: "It is very difficult to develop an X-ray laser".

Introduction: This problem concerns Einstein's coefficients that describe the interaction of atoms with electromagnetic radiation: the spontaneous emission coefficient (A_{21}), and the stimulated absorption and emission coefficients (B_{12} and B_{21}). We aim to derive the relationship between A_{21} and B_{21}, showing its dependence on the energy separation $\Delta E = E_2 - E_1 = h\nu$ of the two levels. Additionally, we discuss the physical significance of A_{21} and explain why the development of an X-ray laser is challenging.

Solution: According to Einstein's theory of radiation:

(i) B_{12}: Coefficient of stimulated absorption from level 1 to 2.

(ii) B_{21}: Coefficient of stimulated emission from level 2 to 1.

(iii) A_{21}: Coefficient of spontaneous emission from level 2 to 1.

In thermal equilibrium, the number of upward transitions equals the number of downward transitions:

$$N_1 B_{12} \rho(\nu) = N_2 B_{21} \rho(\nu) + N_2 A_{21}$$

Here, $\rho(\nu)$ is the spectral energy density of radiation at frequency ν. From Boltzmann distribution:

$$\frac{N_2}{N_1} = \frac{g_2}{g_1} \exp\left(-\frac{h\nu}{kT}\right)$$

where g_1 and g_2 are the degeneracies of levels 1 and 2 respectively.

Substitute $N_2 = N_1 \frac{g_2}{g_1} \exp(-h\nu/kT)$ into the equilibrium equation:

$$N_1 B_{12} \rho(\nu) = N_1 \frac{g_2}{g_1} \exp\left(-\frac{h\nu}{kT}\right) B_{21} \rho(\nu) + N_1 \frac{g_2}{g_1} \exp\left(-\frac{h\nu}{kT}\right) A_{21}$$

Dividing both sides by N_1 and rearranging:

$$\rho(\nu) \left[B_{12} - \frac{g_2}{g_1} B_{21} \exp\left(-\frac{h\nu}{kT}\right)\right] = \frac{g_2}{g_1} A_{21} \exp\left(-\frac{h\nu}{kT}\right)$$

Solving for $\rho(\nu)$:

$$\rho(\nu) = \frac{\frac{g_2}{g_1} A_{21}}{B_{12} \exp\left(\frac{h\nu}{kT}\right) - \frac{g_2}{g_1} B_{21}}$$

This must match Planck's radiation formula:

$$\rho(\nu) = \frac{8\pi h\nu^3}{c^3} \frac{1}{\exp\left(\frac{h\nu}{kT}\right) - 1}$$

Comparing the denominators, we require:

$$B_{12} = \frac{g_2}{g_1} B_{21}$$

This relationship follows from detailed balance considerations and quantum mechanical selection rules. For non-degenerate levels ($g_1 = g_2$), we have $B_{12} = B_{21}$.

Comparing the numerators:

$$\frac{\frac{g_2}{g_1} A_{21}}{B_{12}} = \frac{8\pi h\nu^3}{c^3}$$

Using $B_{12} = \frac{g_2}{g_1} B_{21}$:

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}$$

This clearly shows that the ratio depends on the cube of the transition frequency ν, or equivalently on the cube of the energy separation $\Delta E = h\nu$:

$$\frac{A_{21}}{B_{21}} \propto \nu^3 \propto (\Delta E)^3$$

Physical Significance of A_{21}: The coefficient A_{21} represents the probability per unit time that an atom in the excited state 2 will spontaneously decay to the lower state 1, emitting a photon of energy $h\nu$. This process does not require external radiation and is responsible for phenomena like fluorescence and thermal radiation. It determines the natural linewidth and lifetime of excited states.

Justification: Difficulty in Developing X-ray Lasers:

Due to the dependence:

$$A_{21} \propto \nu^3$$

For X-ray transitions, ν is extremely large (typically 10^{18} Hz or higher), hence A_{21} becomes extremely high. This implies:

(a) The spontaneous decay of excited X-ray states is extremely fast (lifetimes $\sim 10^{-15}$ seconds or less).

(b) For laser action, we need population inversion: $N_2 > N_1$. However, the rate of spontaneous decay from level 2 is $N_2 A_{21}$, which becomes enormous for X-ray frequencies.

(c) The pumping rate required to maintain population inversion must exceed this spontaneous decay rate, requiring extremely high power densities that are difficult to achieve and maintain.

(d) The stimulated emission cross-section is proportional to B_{21}, but since $A_{21}/B_{21} \propto \nu^3$, the relative importance of spontaneous emission over stimulated emission increases dramatically at X-ray frequencies.

(e) Consequently, achieving and maintaining the necessary population inversion for lasing action in the X-ray region becomes extremely challenging.

Conclusion: The ratio $\frac{A_{21}}{B_{21}}$ scales with the cube of the transition frequency, indicating strong dependence on energy level separation. The A_{21} coefficient quantifies spontaneous emission probability, critical in determining excited state lifetimes and natural linewidths. The extremely high spontaneous emission rates at X-ray frequencies make it nearly impossible to establish and maintain the population inversion necessary for laser operation, making X-ray laser development one of the most challenging problems in laser physics.

A/P

68 Derive an expression for intermodal dispersion for a multimodal step-index fibre

Introduction: In a multimode step-index fiber, different modes of light travel along different paths. This difference in path length leads to a variation in the time taken for these modes to travel the length of the fiber, a phenomenon known as intermodal dispersion. This causes a broadening of the transmitted light pulse. We will derive an expression for this time difference (Δt) between the fastest and slowest modes.

Derivation: Let's consider a step-index fiber with the following parameters:

- n_1: refractive index of the core
- n_2: refractive index of the cladding ($n_1 > n_2$)
- L: length of the fiber
- c: speed of light in vacuum

1. Minimum Transit Time (t_{min})

The fastest mode is the axial ray, which travels straight down the center of the core without any reflections. Its path length is simply L. The speed of light in the core is $v = c/n_1$. Therefore, the minimum time taken is:

$$t_{\text{min}} = \frac{L}{v} = \frac{n_1 L}{c}$$

2. Maximum Transit Time (t_{max})

The slowest mode corresponds to the ray that propagates at the maximum acceptance angle and undergoes total internal reflection at the core-cladding boundary. For total internal reflection at the core-cladding interface, the condition is:

$$n_1 \sin\phi_c = n_2$$

where ϕ_c is the critical angle of incidence at the core-cladding boundary.

Let θ be the angle the ray makes with the fiber axis inside the core. The relationship between the angle with the axis (θ) and the angle of incidence at the core-cladding boundary (ϕ) is: $\phi = 90 - \theta$.

At the critical condition: $\phi = \phi_c$, so $\theta_c = 90 - \phi_c$.

Therefore: $\cos\theta_c = \cos(90 - \phi_c) = \sin\phi_c = \frac{n_2}{n_1}$

The actual path length for this ray traveling in a zigzag pattern is:

$$L' = \frac{L}{\cos\theta_c} = \frac{L}{n_2/n_1} = \frac{n_1 L}{n_2}$$

The time taken for this slowest mode to travel the length of the fiber is:

$$t_{\text{max}} = \frac{L'}{v} = \frac{n_1 L/n_2}{c/n_1} = \frac{n_1^2 L}{c n_2}$$

3. Intermodal Dispersion (Δt)

The total intermodal dispersion is the difference between the maximum and minimum transit times:

$$\Delta t = t_{\text{max}} - t_{\text{min}} = \frac{n_1^2 L}{c n_2} - \frac{n_1 L}{c}$$

Factoring out $\frac{n_1 L}{c}$:

$$\Delta t = \frac{n_1 L}{c}\left(\frac{n_1}{n_2} - 1\right) = \frac{n_1 L}{c}\left(\frac{n_1 - n_2}{n_2}\right)$$

4. Weak Guidance Approximation

For most optical fibers, the refractive index difference between the core and cladding is very small (weak guidance). We define the relative refractive index difference, Δ, as:

$$\Delta = \frac{n_1 - n_2}{n_1} \ll 1$$

From this definition: $n_1 - n_2 = n_1\Delta$ and $n_2 = n_1(1-\Delta)$.

Substituting into our exact expression:

$$\Delta t = \frac{n_1 L}{c}\left(\frac{n_1\Delta}{n_1(1-\Delta)}\right) = \frac{n_1 L\Delta}{c(1-\Delta)}$$

For small Δ, $(1-\Delta)^{-1} \approx 1 + \Delta \approx 1$, so:

$$\Delta t \approx \frac{n_1 L\Delta}{c}$$

This can also be expressed in terms of the numerical aperture (NA), where $NA = \sqrt{n_1^2 - n_2^2}$.

We know that:

$$NA^2 = n_1^2 - n_2^2 = (n_1 - n_2)(n_1 + n_2) = n_1\Delta(n_1 + n_2)$$

Under the weak guidance approximation, $n_1 + n_2 \approx 2n_1$, so:

$$NA^2 \approx n_1\Delta \cdot 2n_1 = 2n_1^2\Delta$$

This gives $\Delta \approx \frac{(NA)^2}{2n_1^2}$. Substituting this into our approximate expression:

$$\Delta t \approx \frac{n_1 L}{c}\left(\frac{(NA)^2}{2n_1^2}\right) = \frac{L(NA)^2}{2n_1 c}$$

Conclusion: The time delay due to intermodal dispersion in a step-index fiber is given by:

$$\Delta t = \frac{n_1 L}{c}\left(\frac{n_1 - n_2}{n_2}\right)$$

Under the common weak guidance approximation ($\Delta \ll 1$), this simplifies to:

$$\Delta t \approx \frac{n_1 L\Delta}{c} \quad \text{or} \quad \Delta t \approx \frac{L(NA)^2}{2n_1 c}$$

This shows that pulse broadening is directly proportional to the fiber length and is dependent on the refractive index profile of the fiber.

69 A pulse of $\lambda_0 = 600$ nm and $\Delta\lambda = 10$ nm propagates through a fibre which has a material dispersion coefficient of 50 ps per km per nm at 600 nm. Calculate the pulse broadening in traversing a 10 km length of the fibre. If the pulse width at the input of the fibre is 12 ns, what will be the pulse width at the output of the fibre?

Introduction: The problem concerns the temporal broadening of an optical pulse due to chromatic (material) dispersion in an optical fibre. The parameters provided are:

- Central wavelength of the pulse: $\lambda_0 = 600\,\text{nm}$
- Spectral width (FWHM) of the pulse: $\Delta\lambda = 10\,\text{nm}$
- Material dispersion coefficient: $D = 50\,\text{ps/km/nm}$ at λ_0
- Fibre length: $L = 10\,\text{km}$
- Initial pulse width: $\Delta t_{\text{in}} = 12\,\text{ns}$

We are to calculate the pulse broadening $\Delta t_{\text{broadening}}$ due to dispersion and then determine the output pulse width Δt_{out} after propagation.

Solution:

Pulse broadening due to chromatic dispersion over a distance L is given by:

$$\Delta t_{\text{broadening}} = |D| \cdot \Delta\lambda \cdot L$$

Substituting the given values:

$$\Delta t_{\text{broadening}} = 50\,\text{ps/km/nm} \cdot 10\,\text{nm} \cdot 10\,\text{km} = 5000\,\text{ps}$$

Converting to nanoseconds:

$$\Delta t_{\text{broadening}} = \frac{5000\,\text{ps}}{1000} = 5\,\text{ns}$$

Assuming Gaussian pulse shapes, the output pulse width is related to the input width and the broadening by:

$$\Delta t_{\text{out}} = \sqrt{\Delta t_{\text{in}}^2 + \Delta t_{\text{broadening}}^2}$$

Substituting values:

$$\Delta t_{\text{out}} = \sqrt{(12\,\text{ns})^2 + (5\,\text{ns})^2} = \sqrt{144 + 25}\,\text{ns} = \sqrt{169}\,\text{ns} = 13\,\text{ns}$$

Conclusion: The pulse broadening due to chromatic dispersion over 10 km is $5\,\text{ns}$. The output pulse width after the fibre is $13\,\text{ns}$.

70 Calculate the minimum thickness of a quartz plate which would behave as a quarter-wave plate for wavelength of light, $\lambda = 6000$ Å. The refractive indices for ordinary and extraordinary rays are $\mu_o = 1.544$ and $\mu_e = 1.553$.

Introduction: A quarter-wave plate introduces a phase difference of $\frac{\pi}{2}$ (or $\lambda/4$) between the ordinary and extraordinary rays due to birefringence. We are given:

- Wavelength of light: $\lambda = 6000\,\text{Å} = 600\,\text{nm}$
- Refractive index for ordinary ray: $\mu_o = 1.544$
- Refractive index for extraordinary ray: $\mu_e = 1.553$

We are to calculate the minimum thickness t of the quartz plate that results in a path difference of $\lambda/4$ between the two rays.

Solution:

The optical path difference Δ introduced by a birefringent plate of thickness t is given by:

$$\Delta = (\mu_e - \mu_o) \cdot t$$

For a quarter-wave plate, we require:

$$\Delta = \frac{\lambda}{4}$$

Equating the two expressions:

$$(\mu_e - \mu_o) \cdot t = \frac{\lambda}{4}$$

Solving for t:

$$t = \frac{\lambda}{4(\mu_e - \mu_o)} = \frac{600\,\text{nm}}{4(1.553 - 1.544)} = \frac{600}{4 \cdot 0.009}\,\text{nm}$$

$$t = \frac{600}{0.036}\,\text{nm} \approx 16666.7\,\text{nm}$$

Converting to micrometers:

$$t \approx 16.67\,\mu\text{m}$$

Conclusion: The minimum thickness of the quartz plate required to act as a quarter-wave plate at $\lambda = 600\,\text{nm}$ is approximately $16.67\,\mu$m.